Interlude: ∞-categories

(A) Motivation & General idea

(B) Modeling

(C) Examples

(D) Transfer concepts

(A) Motivation: A homotopy-theoretic point of view

X topological space, $x \in X$

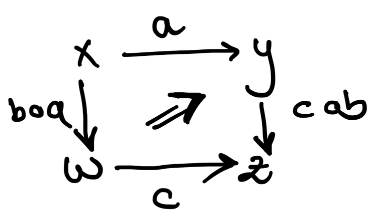
Set/Group theory:	Category theory	higher category theory
Homotopy groups	fundam. groupoid	fundam. n-groupoid

$\pi_0(X)$, $\pi_1(X,x)$ } $\pi_{\leq 1}(X)$: objects: $x \in X$; morphisms: (relative homotopy classes of) paths

$\pi_2(X,x)$ … $\pi_n(X,x)$ } $\pi_{\leq n}(X)$: 2-morphisms: homotopies of paths; ⋮ n-morphisms: homotopy classes of (n-1)-morphisms

fundamental ∞-groupoid: $\pi_\infty(X)$

Composition of paths:



is associative only up to homotopy!

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General idea behind ∞-categories:

An ∞-category should consist of

- objects
- morphisms
- composition of morphisms (associative & unital) — might not be unique in classical sense

\+

- 2-morphisms between morphisms (chain homotopies / natural isomorphisms)
- 3-morphisms

⋮

that are additionally invertible in a certain sense

Slogan: ∞-categories provide a framework to deal with weaker notions of "being equal"

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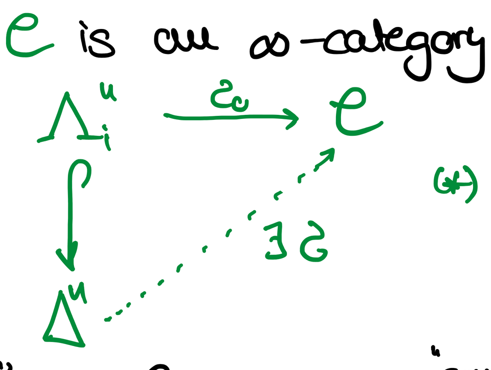
(B) A model of ∞-categories

Idea: Model ∞-categories by simplicial sets

Q: Which class of simplicial sets represents a suitable model?

Main definition:

A simplicial set $\mathcal{C}$ is an ∞-category if for $n>0$ and $0<i<n$



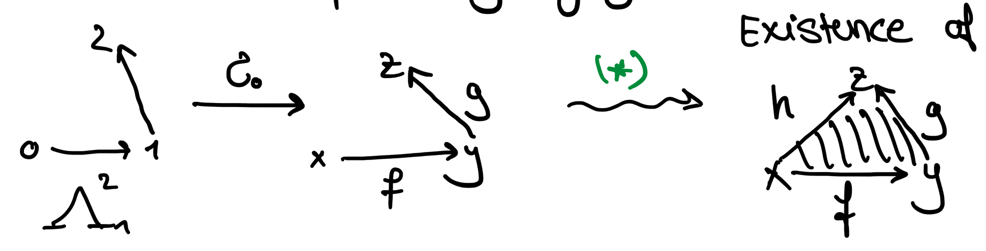
"horn-filler condition"

Objects: $c_0 \ni x: \Delta^0 \longrightarrow \mathcal{C}$

Morphisms: $\mathcal{C}_1 \ni f: x \longrightarrow y$ with $x := d_1 f$ "source" and $y := d_0 f$ "target"

$id_x := s_0(x): x \longrightarrow x$

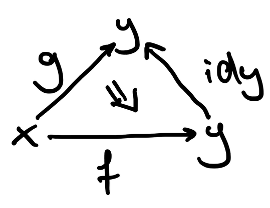
Compositions: $f: x \longrightarrow y$, $g: y \longrightarrow z$



Existence of $\sigma \in \mathcal{C}_2$ corresponds to $h := d_1\sigma: x \longrightarrow z$ "a composition of f and g"

Composition is determined up to homotopy.

Homotopy: Two morphisms $f, g: x \longrightarrow y$ are homotopic ($f \simeq g$) if there is a 2-simplex σ



called homotopy from g to f: $\sigma: g \longrightarrow f$

Proposition:

a) "Being homotopic" is an equivalence relation

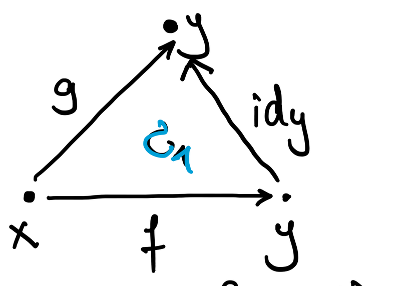
b) Composition is determined up to homotopy.

c) composition is associative & unital up to homotopy.

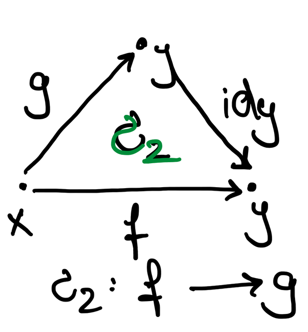
Higher morphisms:

2-morphisms = homotopies are invertible up to

3-morphisms = 2-homotopies are invertible up to …

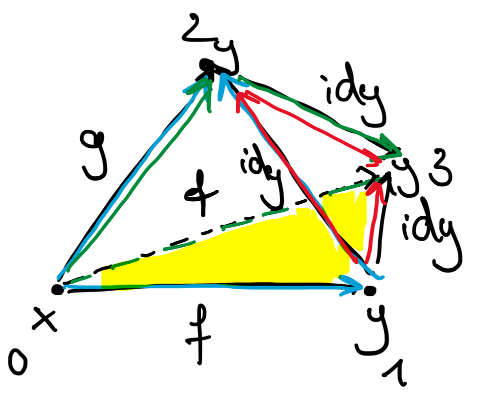


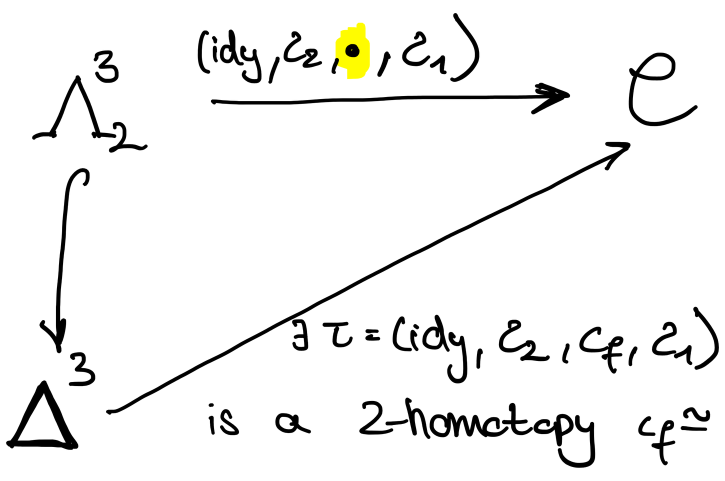
$\sigma_1: g \to f\ (g \simeq f)$



$\sigma_2: f \longrightarrow g\ (f \simeq g)$

id_y





is a 2-homotopy $c_f \simeq \sigma_1 \circ \sigma_2$

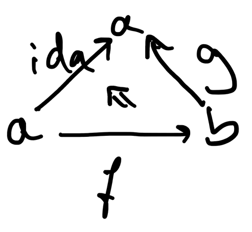
c_f: constant f

⋮ similar observations for higher morphisms

⇒ all higher morphisms are invertible

Proposition: For $x, y \in \mathcal{C}$, the sets of n-morphisms of x and y define a simplicial set "Hom space" $Hom_{\mathcal{C}}(x,y) \in sSet$ which is an ∞-category, s.t. all its morphisms are invertible. (1-morphisms $\in Hom$ ≙ 2-morphisms $\in \mathcal{C}$)

Reminder: A morphism $f: a \longrightarrow b \in \mathcal{C}_1$ is invertible (or isomorphism, equivalence) if $\exists g: b \longrightarrow a \in \mathcal{C}_1$ s.t. $g \circ f \simeq id_a$ and $f \circ g \simeq id_b$



Definition: An ∞-category in which all morphisms are invertible is called an ∞-groupoid. (or Kan complex, space, anima) — space comes from topological spaces

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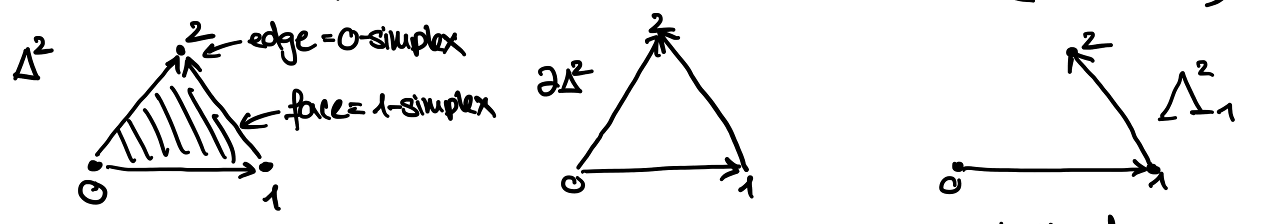
Recap on simplicial sets:

$X \in sSet := Fun(\Delta^{op}, Set)$, $X_n := X([n])$ "n-simplex"

$X_0 \; \overset{d_0}{\underset{d_1}{\leftleftarrows}} \; X_1 \; X_2 \; \dots\dots$

▷ $\Delta^n := Hom(-,[n]) \supseteq \partial\Delta^n \supseteq \Lambda^n_k$ $(0 \leq k \leq n)$



standard simplex, boundary, k-th horn

▷ Yoneda: $X_n \cong Hom_{sSet}(\Delta^n, X)$

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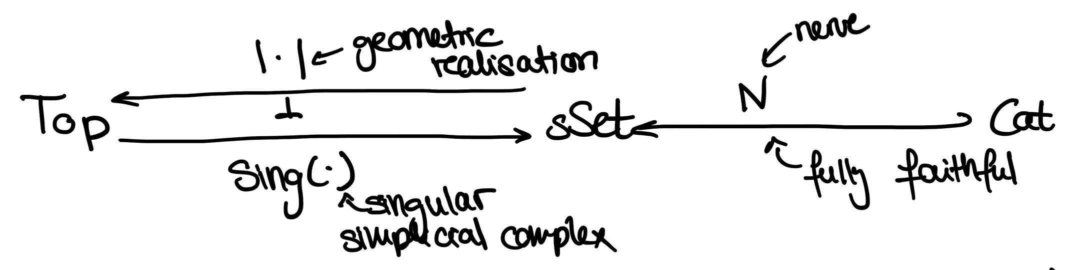
Definition: A morphism $X \longrightarrow Y$ of simplicial sets is a weak equivalence if $|X| \longrightarrow |Y|$ is a weak equiv. in Top.

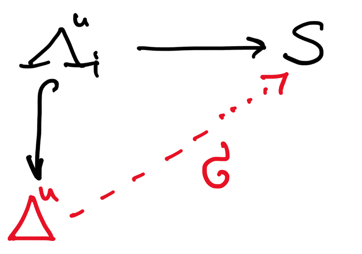


$X \in Top$: $S := Sing(X)$ satisfies

For every $n>0$ and $0 \leq i \leq n$, any map

$\sigma_0: \Lambda^n_i \longrightarrow S$

can be extended to



$\mathcal{C} \in Cat$: $S := N(\mathcal{C})$ satisfies

For every $n>0$ and $0<i<n$ any map can be ~~uniquely~~ extended to

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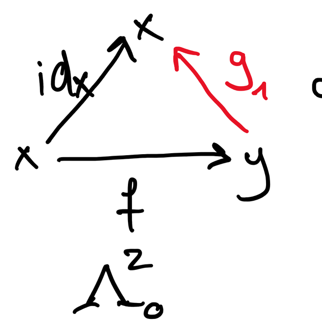
Proposition:

a) Every $Sing(X)$ is an ∞-groupoid.

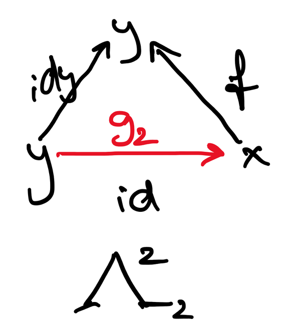
b) If $\mathcal{C}$ is a groupoid, then $N(\mathcal{C})$ is an ∞-groupoid.

Sketch:

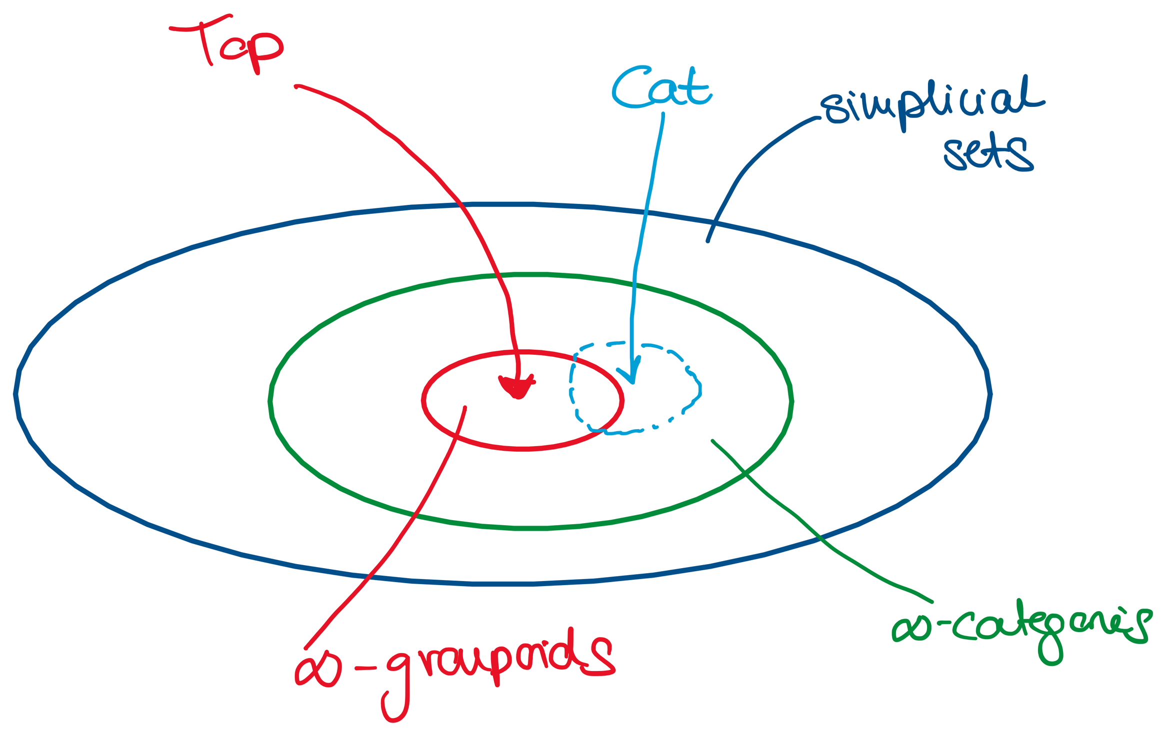
a) Every horn



and



has extension



Definition: $\mathcal{C}$ ∞-category

The homotopy category $Ho(\mathcal{C})$ is a category with

objects = objects of $\mathcal{C}$

morphisms = homotopy classes of morphisms in $\mathcal{C}$

(equivalences are isomorphisms in $Ho(\mathcal{C})$)

⇒ $Ho(N(\mathcal{C})) = \mathcal{C}$

$Ho(Sing(X)) = \pi_{\leq 1}(X)$

"$Sing(X)$ is a model of $\pi_\infty(X)$"

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(C) Examples of ∞-categories

Example:

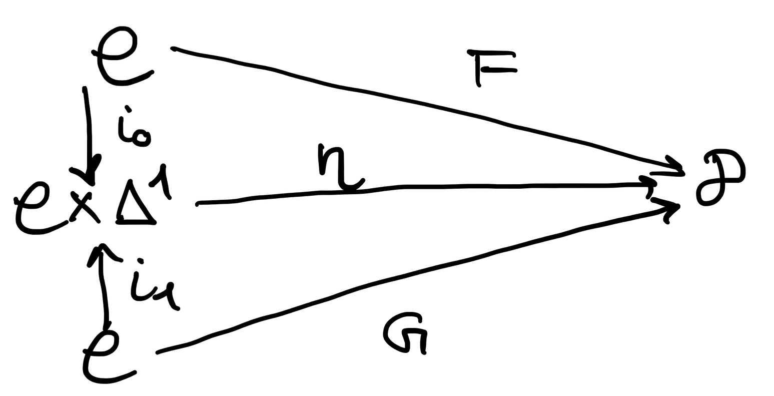
For $\mathcal{C}, \mathcal{D}$ ∞-categories, we define

$$\mathrm{Fun}(\mathcal{C},\mathcal{D}) := \big([n] \longmapsto \mathrm{Hom}_{sSet}(\Delta^n \times \mathcal{C}, \mathcal{D})\big)$$

($[\mathcal{C},\mathcal{D}]$ internal hom sSet)

i.e.

- an ∞-functor $F: \mathcal{C} \to \mathcal{D}$ is just a morphism of simplicial sets
- a natural transformation $\eta: F \to G$ is a simplicial homotopy



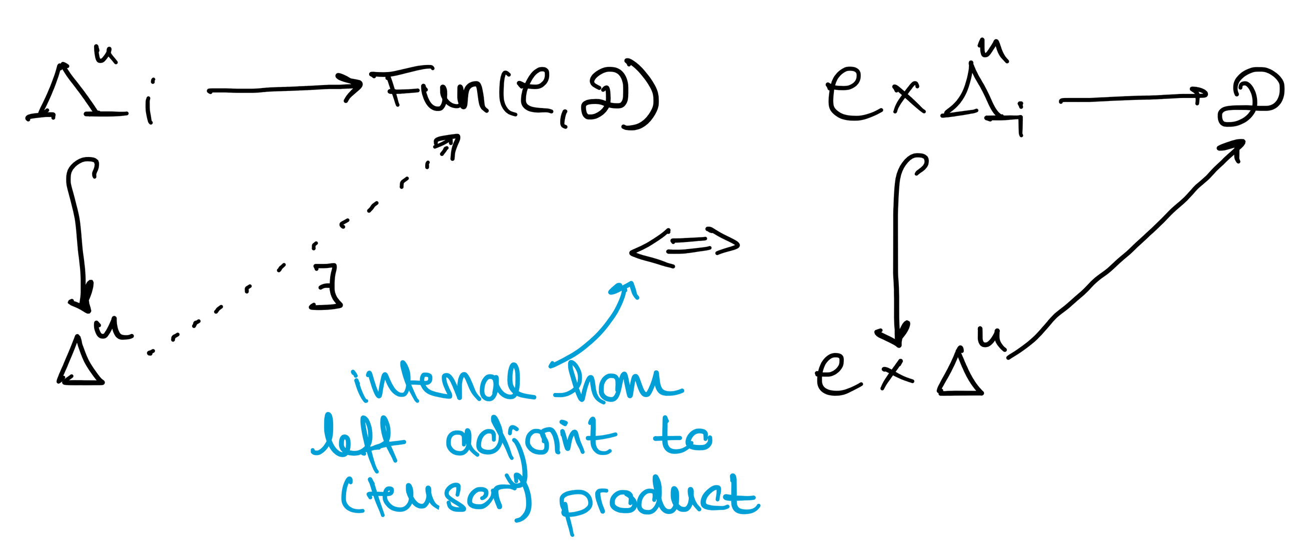
$\eta(c,0) = F(c)$

$\eta(c,1) = G(c)$

⋮

This defines an ∞-category (enough $\mathcal{D}$ ∞-cat)

Idea:



Example: $\mathcal{C}$ ordinary category, $W \subseteq \mathrm{Mor}(\mathcal{C})$ class of weak equivalences

e.g.

$\mathcal{C} = Top$, $W = \{$weak homotopy equivalences$\}$

$\mathcal{C} = sSet$, $W = \{$weak equivalences (by 1.1)$\}$

$\mathcal{C} = Ch(\mathcal{A})$, $W = \{$quasi-isomorphisms$\}$

(chain complexes; $\mathcal{A}$ abelian category)

One can construct ∞-categorical localisation $L(\mathcal{C})$ satisfying universal property:

There is an embedding

$N(\mathcal{C}) \hookrightarrow L(\mathcal{C})$

s.t. weak equivalences are sent to equivalences (∞-categ. sense) and $L(\mathcal{C})$ is universal with this property.

- $L(Top) = L(sSet) =: Ani$ (= ∞-Grpds)
  "∞-category of anima/∞-groupoids/spaces"
  - ▷ objects $\hat{=}$ ∞-groupoids
  - ▷ $\mathrm{Hom}_{\mathcal{C}}(x,y) \in Ani$, i.e. replacement of Set

    $Set \rightleftarrows Ani$ (π_0)
  - ▷ $Ho(Ani) = Ho(Top)$ "homotopy category of spaces"
- $L(Ch(\mathcal{A})) =: \mathbb{D}(\mathcal{A})$ "derived ∞-category"
  - ▷ $Ho(\mathbb{D}(\mathcal{A})) = \mathcal{D}(\mathcal{A})$ "derived category"

(E) Translate categorical concepts

▷ Replace $\mathrm{Hom}_{\mathcal{C}}(x,y) \in Set$ by $\mathrm{Hom}_{\mathcal{C}}(x,y) \in Ani$ and "isomorphism of Sets" by "equivalence in Ani" (weak equivalence in sSet)

e.g.
- Yoneda lemma
- adjoint functors
- fully faithful

▷ Replace "[ ] is determined uniquely up to isomorphism" by "[ ] is determined up to a contractible choice"

e.g.
- all choices of compositions of morphisms form a contractible simplicial set
- $x \in \mathcal{C}$ is initial\terminal in $\mathcal{C}$ if for all $y \in \mathcal{C}$

  $\mathrm{Hom}_{\mathcal{C}}(x,y) \setminus \mathrm{Hom}_{\mathcal{C}}(y,x)$

  is contractible

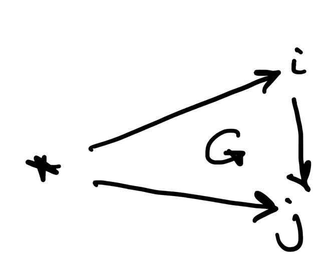
▷ Allow triangles to commute up to homotopy

Example: Limits

Ordinary categories: $F: I \to \mathcal{C}$ diagram of $\mathcal{C}$

Reformulation universal property of the limit of F:

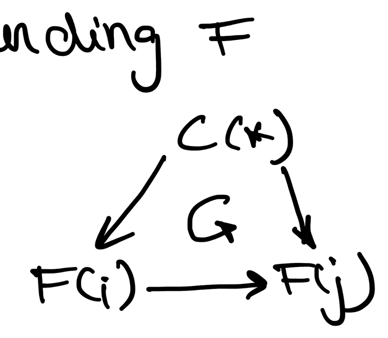
▷ Add initial object to I: "left cone" $I^{\triangleleft}$



s.t. $\mathrm{Hom}_{I^{\triangleleft}}(*,i) = *$ (unique morphism up to isomorph.)

$\mathrm{Hom}_{I^{\triangleleft}}(i,*) = \emptyset$

A cone to F is a diagram $C: I^{\triangleleft} \to \mathcal{C}$ extending F



$\mathcal{C}_{/F}$ = category of cones to F

Then: A limit of F is a terminal object in $\mathcal{C}_{/F}$

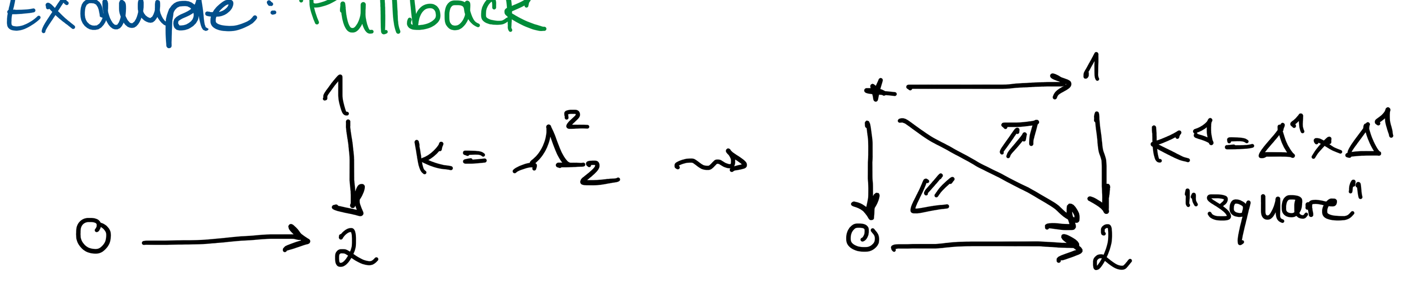
Translation to ∞-categories:

$F: K \to \mathcal{C}$, K simpl. set, $\mathcal{C}$ ∞-category

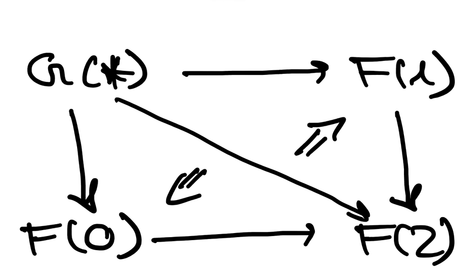
- can define corresponding notions of $K^{\triangleleft}$ and $\mathcal{C}_{/F}$ respecting structures of simpl. sets and ∞-categories

Slogan: Allow all triangles to commute up to homotopy

Example: Pullback

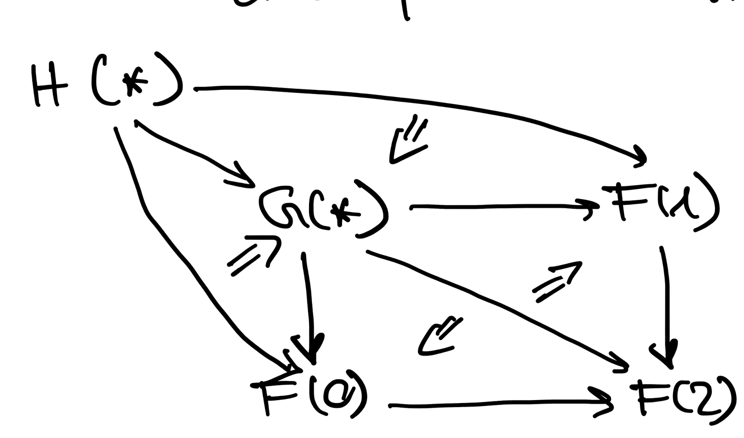


A cone $G: \Delta^1 \times \Delta^1 \to \mathcal{C}$ to F



is a pullback of F if for all $H \in \mathcal{C}_{/F}$

there is a (up to a contractible choice) unique natural transformation $H \to G$, s.t.



i.e. G is terminal in $\mathcal{C}_{/F}$