NOTES ON THE h-TOPOLOGY

ARIYAN JAVANPEYKAR

These are notes written for my talk (26.10.2021) at the seminar on Bhatt-Scholze's paper [BS17] on the affine Witt Grassmannian in Mainz. The topic of my talk is the h-topology, and focuses on Proposition 2.8 in [BS17]. The latter is a characterization of h-sheaves (i.e., set-valued sheaves for the h-topology on the category of finitely presented S-schemes with S a qcqs scheme).

Inaccuracies are all mine of course. Comments and questions more than welcome.

1. Universal submersions

Following Voevodsky [Voe96], we introduce the following terminology.

Definition 1.1. A continuous map of topological spaces $f : X \to Y$ is a *submersion* if it is surjective and Y has the quotient topology, i.e., a subset $U \subset Y$ of Y is open if and only if $f^{-1}(U)$ is open in X.

Example 1.2. If $f : X \to Y$ is an open and surjective continuous map of topological spaces, then f is a submersion. Similarly, if $f : X \to Y$ is a closed and surjective continuous map of topological spaces, then f is a submersion.

Definition 1.3. A morphism of schemes $f : X \to Y$ is a *submersion (in the Zariski topology)* if the underlying map of topological spaces $|X| \to |Y|$ is a submersion. A morphism of schemes $f : X \to Y$ is a *universal submersion* if, for every morphism of schemes $T \to Y$, the morphism $f_T : X \times_Y T \to T$ is a submersion.

Note that a (universal) submersion is surjective (by definition). Moreover, the base change of a universal submersion is a universal submersion. Furthermore, the composition of two universal submersions is a universal submersion.

We start with a very silly example.

Example 1.4 (Silly)**.** Let $f : X \to Y$ be a morphism of schemes. Then $X \sqcup Y \to Y$ is a universal submersion. Indeed, if $U \subset Y$ is a subset, then its inverse image is given by $f^{-1}U \sqcup U$. If its inverse image $f^{-1}U \sqcup U$ is open, then obviously U is open, as required. (This example actually led to some confusion during the talk.)

Example 1.5. If $f : X \to Y$ is an fppf morphism of schemes, then f is a universal submersion. Indeed, recall that fppf morphisms are universally open (i.e., for every morphism of schemes $T \to Y$, the morphism $X \times_Y T \to T$ is open); see [Sta15, 01UA]. Since f is surjective, the claim follows from Example 1.2

Example 1.6. If $f : X \to Y$ is a proper surjective morphism, then f is a universal submersion. Indeed, proper morphisms are universally closed, so that the claim follows from Example 1.2.

How would one construct a non-submersive morphism of schemes? Well, one has to start by avoiding flat morphisms, so one could consider blow-ups. But these are proper and surjective. The idea is then to simply remove points from a blow-up to force non-properness.

Example 1.7. Let k be a field. Let X' be the blow-up of $\mathbb{A}^2_k$ in its origin. Let $E \cong \mathbb{P}^1_k$ be the exceptional locus of $X' \to \mathbb{A}^2$. Let C be a smooth irreducible curve in $\mathbb{A}^2$ passing through the origin, and let $\tilde{C}$ be its (unique) lift to X'. Note that the intersection of $\tilde{C}$ and E consists of precisely one point, say e. Define $X := X' \setminus \{e\}$ and $Y := \mathbb{A}^2_k$. We claim that the morphism $f : X \to Y$ is not a submersion. Indeed, consider the subset $U := C \setminus \{0\}$ in $Y = \mathbb{A}^2$. Note that U is not closed. However, its pull-back $f^{-1}U$

to X is given by $\tilde{C} \setminus \{e\}$. This is closed in X, as it equals $\tilde{C} \cap X$ and $\tilde{C}$ is closed in X'. This proves the claim. (More generally, if $Z \subset Y$ is a closed subset and U is a dense open of the blow-up $\mathrm{Bl}_Z(Y)$ of Y along Z, then the morphism $U \to Y$ is a submersion if and only if $U = \mathrm{Bl}_Z(Y)$. This is proven in a similar fashion; see [Voe96] for details.)

A simpler (and possibly simplest) example (with non-integral schemes) of a surjective non-submersive morphism is the following:

Example 1.8. The morphism $X := \mathbb{A}^1 \setminus \{0\} \sqcup \{0\} \to \mathbb{A}^1$ is not a submersion. Indeed, the inverse image of $\mathbb{A}^1 \setminus \{0\}$ is closed, whereas $\mathbb{A}^1 \setminus \{0\}$ is not closed in $\mathbb{A}^1$. (Or: the inverse image of the non-open point $0 \in \mathbb{A}^1$ in X is open.)

2. Valuation rings (briefly)

We recall the basic definitions and give some relevant examples.

Definition 2.1. Let K be a field, and let $A, B \subset K$ be local rings contained in K. We say that B *dominates* A if $A \subset B$ and $m_A = m_B \cap A$. We say that a ring A is a *valuation ring* if A is a local domain and A is maximal for the relation of domination among local rings contained in $\mathrm{Frac}(A)$.

The following two lemmas give a useful characterisation of valuation rings; see [Sta15, 00I8]

Lemma 2.2. *Let A be a valuation ring with fraction field K. Let $x \in K^\times$. Then $x \in A$ or $x^{-1} \in A$.*

Lemma 2.3. *A subring A of a field K such that, for all x in $K^\times$, we have that either $x \in A$ or $x^{-1} \in A$ (or both) is a valuation ring with fraction field K.*

Note that every field is a valuation ring (with this definition), and that discrete valuation rings are valuation rings. In fact, if A is a *noetherian* valuation ring, then A is either a field or a discrete valuation ring. Examples of non-noetherian valuation rings are ubiquitous. For example, the valuation ring associated to the (extension of the) p-adic valuation on the algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ is a non-noetherian valuation ring.

Definition 2.4. An *extension of valuation rings* is a faithfully flat morphism $V \to W$ of valuation rings. Equivalently, an injective local homomorphism $V \to W$ of valuation rings.

The main example of an extension of valuation rings in this seminar will be the perfection $V \to V^{\mathrm{perf}}$ of a valuation ring V of characteristic $p > 0$. Non-examples of extensions of valuation rings are $\mathbb{Z}_p \subset \mathbb{Q}_p$ which is (injective, but not local), and the morphism $\mathbb{Z}_p \to \mathbb{Z}/p\mathbb{Z}$ (which is local, but not injective).

3. v-covers

We start with the definition of Bhatt-Scholze [BS17, §2]

Definition 3.1. A morphism of qcqs schemes $f : X \to Y$ is a *v-cover* if, for every valuation ring V and every morphism $\mathrm{Spec}\, V \to Y$, there is an extension of valuation rings $V \to W$ and a morphism $\mathrm{Spec}\, W \to X$ such that the diagram

$$\begin{array}{ccc} \mathrm{Spec}\, W & \longrightarrow & X \\ {\scriptstyle \text{extension}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}\, V & \longrightarrow & Y \end{array}$$

commutes.

Remark 3.2. Note that a v-cover is surjective. Indeed, let $X \to Y$ be a v-cover, and let $y \in Y$ be a point with residue field $k := k(y)$. Note that k is a valuation ring. Then, by definition, there is an extension of valuation rings $k \to W$ and a morphism $\mathrm{Spec}\, W \to X$ lifting the morphism $y : \mathrm{Spec}\, k \to Y$. In particular, if x is the image of $\mathrm{Spec}\, K(W) \to \mathrm{Spec}\, W \to X$, then x lies in the fibre X_y.

The base change of a v-cover is a v-cover (use the universal property of pull-back and the definition of v-covers). Furthermore, the composition of two universal submersions is a universal submersion (use that the composition of extensions of valuation rings is an extension of valuation rings).

Intuitively, a morphism is a v-cover if and only if it is surjective on valuation rings (and not just on points). That is, valuation rings in the target "lift" to valuation rings in the domain.

Definition 3.3. A *refinement* of a v-cover $f : X \to Y$ is a v-cover $X' \to Y$ which factors over f.

To check descent conditions for presheaves in some Grothendieck topology, one may (sometimes) pass to refinements. This will be made more precise below.

We start again with a silly example.

Example 3.4 (Silly)**.** Let $X \to Y$ be a morphism of qcqs schemes. Then the morphism $X \sqcup Y \to Y$ is (quite obviously) a v-cover.

We now show that proper surjective morphisms (resp. fppf morphisms) are v-covers.

Lemma 3.5. *Let $f : X \to Y$ be a surjective universally closed morphism of qcqs schemes. Then f is a v-cover. (Actually, it suffices to assume f is surjective and universally specializing.)*

Proof. Let V be a valuation ring with fraction field K and let $\operatorname{Spec} V \to Y$ be a morphism. To show that there is an extension $V \to W$ and a morphism $\operatorname{Spec} W \to X$ lifting the morphism $\operatorname{Spec} V \to Y$, we may and do assume that $Y = \operatorname{Spec} V$. (Indeed, we can replace f by $X_V \to \operatorname{Spec} V$, as the latter is still surjective and universally closed.) Since $X \to Y$ is surjective (on points), there is a field extension L/K and a morphism $\operatorname{Spec} L \to X$ such that

$$\begin{array}{ccc} \operatorname{Spec} L & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \xrightarrow{\text{generic point}} & \operatorname{Spec} V \end{array}$$

commutes. Let η be the image of $\operatorname{Spec} L \to X$, and note that $f(\eta)$ is the generic point η_V of $\operatorname{Spec} V$. Since $f : X \to Y$ is closed, we have that specializations lift; see [Sta15, 0066]. Thus, the specialization from η_V to the special point s_V of $\operatorname{Spec} V$ lifts to X along f, i.e., there is a point s in X such that η specializes to s in X and $f(s) = s_V$. Now, let W be a valuation ring and let $\operatorname{Spec} W \to X$ be a morphism such that the generic point η_W of $\operatorname{Spec} W$ maps to η and the special point s_W of $\operatorname{Spec} W$ maps to s; such data exists by [Sta15, 01J8]. To conclude the proof, note that the composed morphism $\operatorname{Spec} W \to X \to \operatorname{Spec} V$ is dominant (as its image contains the generic point of $\operatorname{Spec} V$) and that the special point of $\operatorname{Spec} W$ maps to the special point of $\operatorname{Spec} V$ (by construction). In particular, $V \to W$ is injective and local, as required. $\square$

Example 3.6. By Lemma 3.5, a proper surjective morphism is a v-cover. (Note that Lemma 3.5 requires no separatedness nor finite type assumptions.)

Lemma 3.7. *Let $f : X \to Y$ be a suirjective universally generalizing morphism of qcqs schemes. Then f is a v-cover. In particular, faithfully flat (not necessarily locally finitely presented) morphisms are v-covers.*

Proof. The proof is similar to the proof of Lemma 3.5.

Let V be a valuation ring with residue field κ and let $\operatorname{Spec} V \to Y$ be a morphism. As in the proof of Lemma 3.5, we may and do assume that $Y = \operatorname{Spec} V$ to show that there is an extension $V \to W$ and a morphism $\operatorname{Spec} W \to X$ lifting the morphism $\operatorname{Spec} V \to Y$. Now, since $X \to Y$ is surjective (on points), there is a field extension ℓ/κ and a morphism $\operatorname{Spec} \ell \to X$ such that

$$\begin{array}{ccc} \operatorname{Spec} \ell & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} \kappa & \xrightarrow{\text{special point}} & \operatorname{Spec} V \end{array}$$

commutes. Let s be the image of $\operatorname{Spec} \ell \to X$, and note that $f(s)$ is the special point s_V of $\operatorname{Spec} V$. Since $f : X \to Y$ is generalizing by assumption [Sta15, 03HV], there is a point η in X such that s generizes to η in X and $f(\eta) = \eta_V$. Now, let W be a valuation ring and let $\operatorname{Spec} W \to X$ be a morphism such that the generic point η_W of $\operatorname{Spec} W$ maps to η and the special point s_W of $\operatorname{Spec} W$ maps to s; such data exists by [Sta15, 01J8]. To conclude the proof, note that the composed morphism $\operatorname{Spec} W \to X \to \operatorname{Spec} V$ is dominant (as its image contains the generic point of $\operatorname{Spec} V$) and that the special point of $\operatorname{Spec} W$ maps to the special point of $\operatorname{Spec} V$ (by construction). In particular, $V \to W$ is injective and local, as required. □

Example 3.8. By Lemma 3.7, an fppf morphism is a v-cover. Also, by [Sta15, 040F], a surjective universally open morphism is a surjective universally generalizing morphism, and thus a v-cover (by Lemma 3.7).

Remark 3.9 (v-covers are neither generalizing nor specializing in general)**.** Let X be an integral surface over a field k, and let x be a point of codimension one. Then $Y := X \sqcup \{x\} \to X$ is a v-cover. However, this morphism is neither generalizing nor specializing. (Indeed, let η be the generic point of X, so that x generizes to η. Take $s := x$ in the complement of X in Y. Then, s does not generize to any point lying over η. Similarly, let $x_0 \neq x$ be in the closure of $x \in X$, so that x specializes to x_0. Then, there is no point in the closure $\{s\}$ of $\{s\}$ lying over x_0.) Interestingly though, a morphism is a v-cover if and only if it is "universally weakly-generalizing"; see Proposition 4.4.

How would one go about constructing (surjective) morphisms which aren't v-covers? Well, as in the case of universal submersions, the above shows that we have to avoid both proper and flat morphisms.

Example 3.10. (Similar to Example 1.7] Let k be a field. Let X' be the blow-up of $\mathbb{A}^2_k$ in its origin. Let $E \cong \mathbb{P}^1_k$ be the exceptional locus of $X' \to \mathbb{A}^2$. Let C be a smooth irreducible curve in $\mathbb{A}^2$. Define $V = \mathcal{O}_{C,0}$ to be the local ring of C at the origin (in $\mathbb{A}^2$). Let $\tilde{C}$ be the unique lift of C to X' and let e be the unique point of $\tilde{C}$ lying on the exceptional locus of $X' \to \mathbb{A}^2$. Then, the unique lift of $\operatorname{Spec} V \to \mathbb{A}^2_k$ to a morphism $\operatorname{Spec} V \to X'$ sends the special point of $\operatorname{Spec} V$ to e. In fact, if $V \to W$ is an extension and $\operatorname{Spec} W \to X'$ is a lift of $\operatorname{Spec} V \to \mathbb{A}^2$, then the special point of $\operatorname{Spec} W$ maps to e. This shows that $X := X' \setminus \{e\} \to \mathbb{A}^2_k$ is not a v-cover.

As in the section on universal submersions, the simplest example of a surjective morphism which is not a v-cover is given by the following

Example 3.11. The morphism $f : X := \mathbb{A}^1 \setminus \{0\} \sqcup \{0\} \to \mathbb{A}^1 =: Y$ is not a v-cover. Indeed, let $V = \mathbb{O}_{\mathbb{A}^1,0}$ be the local ring of the affine line at 0, and note that this is a discrete valuation ring. Consider the natural morphism $\operatorname{Spec} V \to \mathbb{A}^1$. There is clearly no extension $V \to W$ such that the composed morphism $\operatorname{Spec} W \to \operatorname{Spec} V \to \mathbb{A}^1$ factors over $f : X \to Y$.

4. The subtle difference between v-covers and universal submersions

The way we have presented universal submersions and v-covers, it should be clear that these notions are closely related, and possibly even equivalent to one another. For example, we have the following

Lemma 4.1. *Let $f : X \to Y$ be a morphism of qcqs schemes. If f is a v-cover, then f is a universal submersion.*

Proof. See Proposition 2.13 in [BM21] for a proof. We explain the idea. First, since v-covers are stable by base-change, it suffices to show that f is a submersion.

To show this, let $U \subset Y$ be a subset, and let $V := f^{-1}U$ be its inverse image. Assume that V is open in X, so that V is stable under generization. Let us show that U is stable under generization. Thus, let $s \in U$ and let $\eta \in Y$ specialize to s (so that s generizes to η). Choose a valuation ring V and a morphism $\operatorname{Spec} V \to Y$ such that the generic point of $\operatorname{Spec} V$ maps to η and the special point of $\operatorname{Spec} V$ maps to s.

Since f is a v-cover, there is an extension $V \to W$ and a morphism $\operatorname{Spec} W \to X$ such that

$$\begin{array}{ccc} \operatorname{Spec} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} V & \longrightarrow & Y \end{array}$$

commutes. Now, let η' be the image the generic point of $\operatorname{Spec} W$ in X and let s' be the image of the special point of $\operatorname{Spec} W$ in X. Note that s' lies in V, as $f(s') = s$ lies in U. Moreover, since s' generizes to η' (true?) and V is stable under generization, we conclude that η' lies in V. This implies that $\eta = f(\eta')$ lies in $f(V) = U$, so that U is stable under generization.

We warn the reader that v-covers are not necessarily generalizing or specializing (Remark 3.9). The above line of reasoning does however show that v-covers are "weakly generalizing" and "weakly specializing"; see Remark 4.3 for a brief discussion.

Now, if $X \to Y$ is finitely presented, it follows from Chevalley's theorem that the image U of V is a constructible set (in the Zariski topology). Since a constructible set which is stable under generization is open, this concludes the proof (assuming $X \to Y$ is finitely presented).

If one wants to prove the lemma without making any additional finiteness hypothesis on f, then one has to avoid Chevalley's theorem. Here is how Bhatt-Mathew argue:

First, to prove that U is open in Y, we may and do assume that Y is affine. Since f is surjective (Remark 3.2), it is a quotient for the constructible topology (as any continuous surjection of quasi-compact Hausdorff totally disconnected topological spaces is a quotient map). Therefore, as $V = f^{-1}U$ is Zariski open (hence open in the constructible topology), it follows that U is open in the constructible topology. Since U is stable under generization and open in the constructible topology, it follows that U is open in Y by applying [Sta15, 0903].

Actually, there is also the possibility of invoking [Sta15, 0ETP]. $\square$

Remark 4.2. Rydh defines a morphism to be *universally subtrusive* if after each base-change it is a submersion in the constructible topology and the S-topology. (Being a submersion in the S-topology is the same as being "weakly-specializing".) By Theorem 2.8 in Rydh, a morphism of qcqs schemes is universally subtrusive if and only if it is v-cover. Lemma 4.1 thus implies that a universally subtrusive morphism is a universal submersion. However, one can also prove this directly using the definition. (Details omitted for now.)

Remark 4.3. Let $X \to Y$ be a v-cover (hence surjective). Although f might not generalizing or specializing (Remark 3.9), the proof of Lemma 4.1 shows that, given a point η in Y and a point s in its closure (so that η specializes to s), there **exists** a point s' over s and a point η' over η such that η' specializes to s' (and s' generizes to η'). This is why one might say that f is "weakly-specializing" and "weakly-generalizing". It is interesting to note that a converse statement also holds.

Proposition 4.4. *Let $f : X \to Y$ be a morphism of qcqs schemes. Then f is a v-cover if and only if f is "universally weakly-specializing", i.e., for every scheme $Z \to Y$ and every point η in Z and every s in Z with $s \in \overline{\{\eta\}}$, there exists a point η' in $X \times_Y Z$ and a point s' in $X \times_Y Z$ such that $f(\eta') = \eta, f(s') = s$ and $s' \in \overline{\{\eta'\}}$.*

Proof. (See also [Sta15, 0ETN].) We have already seen in the proof of Lemma 4.1 (as explained in Remark 4.3) that a v-cover is "universally weakly-specializing". Now, assume that for every scheme $Z \to Y$ and every point η in Z and every s in Z with $s \in \overline{\{\eta\}}$, there exists a point η' in $X \times_Y Z$ and a point s' in $X \times_Y Z$ such that $f(\eta') = \eta, f(s') = s$ and $s' \in \overline{\{\eta'\}}$. Let V be a valuation ring and let $\operatorname{Spec} V \to Y$ be a morphism. Then $X_V \to \operatorname{Spec} V$ is "weakly-specializing". Thus, there is a point η lying over η_V and a point s lying over s_V such that η specializes to s in X_V. By the fact that local rings are dominated by valuation rings (see [Sta15,]), there is a valuation ring W and a morphism $\operatorname{Spec} W \to X$ such that η_W is mapped to η and s_W is mapped to s. The compositie $\operatorname{Spec} W \to X_V \to \operatorname{Spec} V$ induces an inclusion $V \to W$ which is an extension of valuation rings (by construction). This shows that f is a v-cover. $\square$

There is a subtle difference between being a universal submersion and being a v-cover, as the following example shows.

Example 4.5 (Example 4.3 in [Ryd10]). Let V be a valuation ring of rank two, and write $\operatorname{Spec} V = \{x_0 \leq x_1 \leq x_2\}$. (This means that x_2 specializes to x_1 and x_1 specializes to x_0.) Then, we may choose elements $s, t \in V$ such that $\operatorname{Spec}(V/s) = \{x_0, x_1\}$ and $\operatorname{Spec}(V_t) = \{x_1, x_2\}$. Let $S' = \operatorname{Spec}(V/s \times V_t) = \operatorname{Spec}(V/s) \sqcup \operatorname{Spec}(V_t)$.

We first note that the natural morphism $S' \to S$ is not a v-cover. Indeed, suppose that $S' \to S$ is a v-cover. Then, there is an extension $V \to W$ and a morphism $\operatorname{Spec} W \to S'$ over S. Since $\operatorname{Spec} W$ is connected, its image in S' is connected. Therefore, $\operatorname{Spec} W \to S'$ factors over either $\operatorname{Spec}(V/s)$ or $\operatorname{Spec}(V_t)$. Since neither $\operatorname{Spec}(V/s)$ nor $\operatorname{Spec}(V_t)$ surject onto $\operatorname{Spec}(V)$ this leads to a contradication. Thus, $S' \to S$ is not a v-cover.

Now, we claim that $S' \to S$ is a finitely presented universal submersion. Clearly, the morphism is finitely presented, thus it suffices to show that $S' \to S$ is a universal submersion. We give two proof sketches.

First, to check that $S' \to S$ is a universal submersion, we may use Theorem 2.8.(ii) of [Ryd10]. This says that (the quasi-compact morphism) $S' \to S$ is universally submersive if (and only if), for any valuation ring A and morphism $\operatorname{Spec} A \to S$ (i.e., morphism $V \to A$), the pull-back $S' \times_S \operatorname{Spec} A \to \operatorname{Spec} A$ is submersive. To show this, we suppose that $V = A$ for simplicity. Let $Z \subset \operatorname{Spec} V$ be one of the eight possible subsets and assume $f^{-1}Z$ is closed in S'. One can now check by hand that Z must be closed. (Details omitted.)

Another possibility is to invoke Ferrand's theorem on push-outs or to invoke [Sta15, 0EU8].

There is however no difference between v-covers and universal submersions when working with noetherian schemes by the following theorem of Rydh.

Theorem 4.6 (Rydh). *Let $f : X \to Y$ be a morphism of qcqs schemes. Assume that Y is **noetherian**. Then f is a v-cover if and only if f is a universal submersion.*

The proof of this theorem is given in [Ryd10]. Example 4.5 shows that the statement is false without the assumption that Y is noetherian.

Remark 4.7. The following result of Rydh and Bhatt-Mathew [BM21, Proposition 2.19] is interesting to note, as it makes no noetherianity assumption: A morphism $X \to Y$ of qcqs schemes is an arc-cover if and only if it is universally spectrally submersive. (We omit definitions; see [BM21].)

5. The h-topology

Let S be a qcqs scheme and let $(\mathrm{Sch}/S)^{\mathrm{fp}}$ be the category of finitely presented schemes over S. Note that each object of this category is a qcqs scheme. The h-topology on $(\mathrm{Sch}/S)^{\mathrm{fp}}$ is the Grothendieck topology generated by finitely presented v-covers.

Remark 5.1. If S is noetherian, then Voevodsky defined the h-topology using finite type universal submersions. By Theorem 4.6, this coincides with the definition of the h-topology above when S is noetherian. In general, by Example 4.5, the topology on $(\mathrm{Sch}/S)^{\mathrm{fp}}$ generated by finitely presented universal submersions does not coincide with the h-topology (as defined above).

We refer to coverings in the h-topology as h-coverings. If a presheaf F on $(\mathrm{Sch}/S)^{\mathrm{fp}}$ is a sheaf for the h-topology, we say that F is an h-sheaf. Our goal is to understand what it means to be an h-sheaf. As the following example shows, the property of being an h-sheaf is quite restrictive, as the "structure sheaf" is not an h-sheaf.

Example 5.2. The sheaf $\mathcal{O} = \operatorname{Hom}_S(-, \mathbb{A}^1_S)$ is not an h-sheaf; see [Sta15, Tag 0EV0]. Indeed, a finitely presented nilimmersion $X \to Y$ is an h-covering, as it is a proper surjective morphism (of finite presentation). Assume that F is an h-sheaf. Then the descent condition for $X \to Y$ is that $F(Y) \to F(X)$ is injective and that its image in $F(X)$ is the equalizer of the two maps from $F(X)$ to $F(X \times_Y X)$. Note that $X \times_Y X = X$. Thus, the equalizer of the two maps from $F(X)$ to $F(X \times_Y X)$ is $F(X)$. Thus, $F(Y) = F(X)$. This equality obviously fails for $\mathcal{O}$. (E.g., X and Y affine and $X \to Y$ not an isomorphism)

Part of this seminar is dedicated to understanding the sheafification of $\mathcal{O}$ when $S = \mathbb{F}_p$. We will see that the h-sheafification of $\mathcal{O}$ is the sheaf $\mathcal{O}_{perf}$ which assigns to each scheme X over $\mathbb{F}_p$ the perfection $\mathcal{O}(X)^{perf}$ of $\mathcal{O}(X)$.

Let S be a qcqs scheme and let F be an h-sheaf on $(\mathrm{Sch}/S)^{\mathrm{fp}}$. The sheaf F has two important properties.

(1) F is an fppf sheaf.
(2) Let Y be an affine scheme of finite presentation over S, let $X \to Y$ be a proper surjective morphism of finite presentation, let $Z \subset Y$ be a finitely presented closed subset with preimage E in X such that $X \to Y$ is an isomorphism over $Y \setminus Z$. Then, the following (commutative) diagram

$$\begin{array}{ccc} F(Y) & \longrightarrow & F(X) \\ \downarrow & & \downarrow \\ F(Z) & \longrightarrow & F(E) \end{array}$$

is a pull-back square (i.e., Cartesian).

Let us prove these two properties. First, note that fppf coverings are h-coverings, so that 1) is clear. To prove 2), we follow the first ("easy") part of the proof of Proposition 2.8 in Bhatt-Scholze [BS17]. Since $X \to Y$ is proper surjective finitely presented, it is an h-covering. Thus, $F(Y)$ is the equalizer of $F(X) \rightrightarrows F(X \times_Y X)$. Note that, $X \sqcup E \times_Z E \to X \times_Y X$ is an h-covering as well. Therefore, we see that

$$\begin{aligned} F(Y) &= \mathrm{eq}(F(X) \rightrightarrows F(X \times_Y X)) \\ &= \mathrm{eq}(F(X) \rightrightarrows F(X \sqcup E \times_Z E)) \\ &= \mathrm{eq}(F(X) \rightrightarrows F(X) \times F(E \times_Z E)) \\ &= \mathrm{eq}(F(X) \rightrightarrows F(E \times_Z E)). \end{aligned}$$

We can now use this description of $F(Y)$ to prove that the diagram above is Cartesian. Details omitted (see the first paragraph of the proof of [BS17, Proposition 2.8]). □

The Main Theorem of this talk is Proposition 2.8 in Bhatt-Scholze [BS17].

Theorem 5.3 (Main Theorem). *Let F be a presheaf on $(\mathrm{Sch}/S)^{fp}$. Then F is an h-sheaf if and only if the following two statements hold.*

(1) F is an fppf sheaf.
(2) Let Y be an affine scheme of finite presentation over S, let $X \to Y$ be a proper surjective morphism of finite presentation, let $Z \subset Y$ be a finitely presented closed subset with preimage E in X such that $X \to Y$ is an isomorphism over $Y \setminus Z$. Then, the following (commutative) diagram

$$\begin{array}{ccc} F(Y) & \longrightarrow & F(X) \\ \downarrow & & \downarrow \\ F(Z) & \longrightarrow & F(E) \end{array}$$

is a pull-back square (i.e., Cartesian).

To prove Theorem 5.3, we will invoke structure theorems for h-coverings. In fact, it turns out that, up to refinement, an h-covering factors into the composition of fppf morphisms and "blow-ups" as in 2) above.

6. Two structure theorems

Our first structure theorem is due to Rydh (in the non-noetherian case).

Theorem 6.1 (Rydh, Voevodsky (noetherian case))**.** *Let $f : X \to Y$ be a finitely presented v-cover, where Y is an affine scheme. Then, there is a refinement $X' \to Y$ of f which factors as a quasi-compact open covering $X' \to Y'$ and a proper surjective morphism $Y' \to Y$ of finite presentation.*

Proof. See [Ryd10]. □

To state the second structure theorem, we will require a definition.

Definition 6.2. Let $f : X \to Y$ be a proper surjective morpism of finite presentation between qcqs schemes. Let $n \geq 0$ be an integer.

(1) We say that f is of *inductive level 0* if f has a refinement $X' \to Y$ which factors as $X' \to Y' \to Y$, where $X' \to Y'$ is proper fppf and $Y' \to Y$ is a finitely presented nilimmersion.

(2) We say that f is of inductive level $\leq n$ if f has a refinement $X' \to Y$ which factors as
$$X' \to X_0 \to Y_0 \to Y,$$
where $X' \to X_0$ is proper fppf, $Y_0 \to Y$ is a finitely presented nilimmersion, and $X_0 \to Y_0$ is a proper surjective morphism of finite presentation which is an isomorphism outside a closed finitely presented subset $Z \subset Y_0$ such that $X_0 \times_{Y_0} Z \to Z$ is of inductive level $\leq n - 1$.

It turns out that every proper surjective morphism is of inductive level $\leq n$ for some n. More precisely, we have the following.

Proposition 6.3. *Let Y be a reduced finite type scheme over $\mathbb{Z}$, and let $f : X \to Y$ be a proper surjective morphism. Then f is of inductive level $\leq \dim(Y)$.*

Proof. We argue by induction on $\dim Y$. First, if $\dim Y = 0$, then Y is the spectrum of a finite étale $\mathbb{F}_p$-algebra. Since "everything" over such a scheme Y is flat, we see that $X \to Y$ is proper fppf, and thus of inductive level 0.

Assume now that $\dim Y > 0$, and suppose that, for every reduced finite type scheme Y' over $\mathbb{Z}$ with $\dim Y' < \dim Y$, a proper surjective morphism $X' \to Y'$ of schemes with Y' reduced and finite type over $\mathbb{Z}$ is of inductive level $\leq \dim Y'$. (Note that we do not need to pass to a refinement here.)

By Raynaud-Gruson's flattening theorem and generic flatness, there is a refinement $X' \to Y$ of $X \to Y$ which factors as $X' \to X_0 \to Y$, where $X' \to X_0$ is proper fppf and $X_0 \to Y$ is a "blow-up", i.e., there is a proper closed reduced subset $Z \subsetneq Y$ such that $X_0 \to Y$ is an isomorphism over $Y \setminus Z$. Now, note that Z is of finite type over $\mathbb{Z}$ and $\dim Z < \dim Y$. Therefore, $X_0 \times_Y Z \to Z$ is of inductive level $\leq \dim Z \leq \dim Y - 1$, as required. □

This immediately gives the following result:

Proposition 6.4. *Let Y be a finite type scheme over $\mathbb{Z}$, and let $f : X \to Y$ be a proper surjective morphism. Then f is of inductive level $\leq \dim(Y)$.*

Proof. Since $X_{red} \to X$ is a v-cover, we may and do assume that X is reduced (as we allow for refinements in the definition of inductive level). In particular, $X \to Y$ factors over $Y_{red} \to Y$. Then, by Proposition 6.3, the morphism $f : X \to Y_{red}$ is of inductive level $\leq \dim Y_{red} = \dim Y$. It follows from the definition of inductive level that $X \to Y$ is of inductive level $\leq \dim Y$ (because we allow for nilimmersions in the target). □

The second structure result we require reads as follows.

Theorem 6.5. *Let $X \to Y$ be a finitely presented proper surjective morphism of qcqs schemes. Then, there is an integer $n \geq 0$ such that f is of inductive level $\leq n$.*

Proof. This follows from Proposition 6.4 and Noetherian approximation. □

We are now ready to prove the Main Theorem.

Proof of Theorem 5.3. We already know that an h-sheaf satisfies conditions 1) and 2). Thus, let F be a presheaf satisfying condition 1) and condition 2). To show that F is an h-sheaf, we first show that F is separated in the h-topology, i.e., for every h-covering $X \to Y$, the map of sets $F(Y) \to F(X)$ is injective. Thus, let a and b be elements of $F(Y)$ which are equal in $F(X)$. To show that $a = b$ we will proceed in several steps.

First, since the h-topology is generated by finitely presented v-covers, we may assume that $X \to Y$ is a finitely presented v-cover.

Next, to check that $a = b$, we may assume that Y is affine. (Here we use that F is an fppf sheaf by assumption. Actually, we only use that F is a separated Zariski sheaf in this step.)

Then, we may (obviously) replace $X \to Y$ by a refinement. Thus, by Rydh-Voevodsky's structure theorem (Theorem 6.1), we may assume that the finitely presented v-cover $X \to Y$ factors as a quasi-compact open covering $X \to Y'$ and a proper surjective finitely presented morphism $Y' \to Y$. Since F is an fppf sheaf, the map of sets $F(Y') \to F(X)$ is injective. Thus, we may assume that $X \to Y$ is a proper surjective finitely presented morphism.

We now invoke our second structure theorem (Theorem 6.5) to see that there is an integer $n \geq 0$ such that $X \to Y$ is of inductive level $\leq n$. To prove $a = b$ in $F(Y)$, we argue by induction n.

If $n = 0$ replacing $X \to Y$ by a refinement if necessary, the morphism $X \to Y$ is a composition of an fppf morphism and a nilimmersion. Since F is an fppf sheaf (and thus separated fppf presheaf) and F satisfies condition (2), it follows that $a = b$ in $F(Y)$.

If $n > 0$, replacing $X \to Y$ by a refinement if necessary, then $X \to Y$ factors as a composition $X \to X_0 \to Y_0 \to Y$ as in the definition of inductive level $\leq n$. Using again that F is an fppf sheaf and condition (2), we may assume that $X = X_0$ and $Y = Y_0$, so that $X \to Y$ is a proper surjective morphism of finite presentation which is an isomorphism outside a closed subset $Z \subsetneq Y$ of finite presentation and such that the proper surjective finitely presented morphism $E := X \times_Y Z \to Z$ is of inductive level $\leq n-1$. By condition (2), we have that $F(Y) = F(X) \times_{F(E)} F(Z)$ and by the induction hypothesis we have that the map of sets $F(Z) \to F(E)$ is injective. This implies that $a = b$. (Indeed, write $a = (a_X, a_Z)$ and $b = (b_X, b_Z)$ and note that $a_{X,E} = a_{Z,E}$ and $b_{X,E} = b_{Z,E}$ by defintion of the pull-back. Moreover, the condition that $a = b$ in $F(X)$ means that $a_X = b_X$. In particular, $a_{X,E} = b_{X,E}$. Thus,

$$a_{Z,E} = a_{X,E} = b_{X,E} = b_{Z,E}.$$

Thus, a_Z and b_Z are equal in $F(E)$. Since $F(Z) \to F(E)$ is injective, this implies that $a_Z = b_Z$. Thus $a = b$ as required.) We conclude that F is separated.

To show that F is a sheaf (i.e., satisfies the descent condition) one argues in a similar way (i.e., reduce to Y affine using fppf sheaf property, apply the two structure theorems to reduce to dealing with proper surjective morphisms of inductive level $\leq n$, and use induction). Details omitted. $\square$

References

[BM21] Bhargav Bhatt and Akhil Mathew. The arc-topology. *Duke Math. J.*, 170(9):1899–1988, 2021.

[BS17] Bhargav Bhatt and Peter Scholze. Projectivity of the Witt vector affine Grassmannian. *Invent. Math.*, 209(2):329–423, 2017.

[Ryd10] David Rydh. Submersions and effective descent of étale morphisms. *Bull. Soc. Math. France*, 138(2):181–230, 2010.

[Sta15] The Stacks Project Authors. *Stacks Project.* http://stacks.math.columbia.edu, 2015.

[Voe96] V. Voevodsky. Homology of schemes. *Selecta Math. (N.S.)*, 2(1):111–153, 1996.

Ariyan Javanpeykar, Institut für Mathematik, Johannes Gutenberg-Universität Mainz, Staudingerweg 9, 55099 Mainz, Germany.

E-mail address: peykar@uni-mainz.de