I Nearby/vanishing cycles for sheaves

X complex manifold

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "\text{special fibre}"'] & X \arrow[d, "f"] \\ 0 \arrow[r] & \mathbb{C} \end{tikzcd}$$

$F \in Sh(X, \mathbb{C})$ - a sheaf

nearby cycles : $\psi_f F \in D(X_0, \mathbb{C})$ + $E : \psi_f F \to \psi_f F$

Virtue : $\psi_f F$ is a complex on X_0 "remembering" $F|_{X_t}$, $0 \neq t \in \mathbb{C}$, $|t| \ll 1$

$$\begin{tikzcd} X_0 \arrow[r, "i"] & X \arrow[d, "f"] & X^* \arrow[l, "j"'] \arrow[d] & \widetilde{X}^* =: X \times_{\mathbb{C}^*} \mathbb{C} \arrow[l, "p"'] \arrow[d] \arrow[dl, phantom, "\square"] \\ & \mathbb{C} & \mathbb{C}^* \arrow[l] & \mathbb{C} \arrow[l] \\ & & e^{2\pi i g} & g \arrow[l, mapsto] \end{tikzcd}$$

$\psi_f F := i^{-1} Rj_* p_* p^{-1} j^{-1} F$. On $\widetilde{X}^*$: $(x, g) \xmapsto{E} (x, g+1)$

Adjunction (E^*, E_*) for sheaves on $\widetilde{X}^*$

$\rightsquigarrow$ adj $p^{-1} j^{-1} F \to E_* E^{-1} p^{-1} j^{-1} F = E_* (p E)^{-1} j^{-1} F$

$= E_* p^{-1} j^{-1} F$

$\rightsquigarrow p_* \text{adj} : p_* p^{-1} j^{-1} F \to p_* E_* p^{-1} j^{-1} F = p_* p^{-1} j^{-1} F$

$\to E := i^{-1} Rj_* p_* \text{adj} : \psi_f F \to \psi_f F$ "monodromy action on $\psi_f F$"

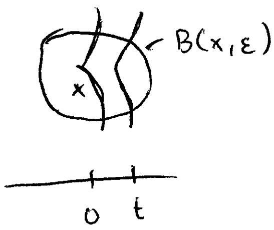
(same letter, different meaning!

$F \to Rj_* p_* p^{-1} j^{-1} F$

$\rightsquigarrow$ $i^{-1} F \to \psi_f F \to \underbrace{\phi_f F}_{} \xrightarrow{+1} =: (\dagger)$

cone of $i^{-1} F \to \psi_f F \in D(X_0, \mathbb{C})$ "vanishing cycles"

<u>Fact 1</u>: $x \in X_0$, $(\psi_f F)_x = R\Gamma(\underbrace{f^{-1}(t) \cap B(x, \varepsilon)}_{\text{Milnor fibre of } f \text{ at } x.}, F)$



$t \neq 0$
$|t| \ll 1$

Fact 2 : X proper $\Rightarrow R\Gamma(X_0, \psi_f) \cong R\Gamma(X_t, F)$ $\quad t \neq 0$, $|t| << 1$.

Apply $R\Gamma(X_0, (\dagger)) \rightsquigarrow H^i(X_0, i^{-1}F) \to H^i(X_t, F_t) \to H^i(X_0, \phi_f) \to H^{i+1}(X_0, i^{-1}F) \to \cdots$

$\phi_f(F)$ measures the discrepancy between the cohomology of X_0 and X_t, $t \neq 0$, $|t| << 1$.

<u>Problem</u>: ψ_f and ϕ_f are hard to compute. $\rightsquigarrow$ construct their D-module counterpart via the Riemann-Hilbert correspondence

$$\begin{tikzcd}
D^b_{reg}(D_X\text{-mod}) \arrow[r, dashed, "?"] \arrow[d, "\wr\, DR"] & D^b_{reg}(D_{X_0}\text{-mod}) \arrow[d, "\wr\, DR"] \\
D^b_c(X, \mathbb{C}) \arrow[r, "\psi_f"'] & D^b_c(X_0, \mathbb{C})
\end{tikzcd}$$

II <u>Nearby and vanishing cycles for D-modules</u>

X complex manifold, D_X = sheaf of finite order diff. operators on $X \subseteq \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$.

$Z \hookrightarrow X$ closed analytic, $\mathcal{J}_Z$: the ideal sheaf of Z.

$V_k(D_X) := \{ P \in D_X \mid \forall \ell \in \mathbb{Z},\ P(\mathcal{J}_Z^{k+\ell}) \subseteq \mathcal{J}_Z^{\ell} \}$ = V-filtration on D_X.

$X = \mathbb{C}^{n+1}$: coordinates $(x_0, \ldots, x_{n-1}, t)$ $\quad Z = \{t = 0\}$ ($\to$ reduce to Z smooth by graph embedding)

$$V_0(D_X) = \Big\{ \sum_{i,k} a_{k,i} (t\partial_t)^i \partial_{x_1}^{k_1} \cdots \partial_{x_{n-1}}^{k_{n-1}} \ \Big|\ a_{k,i} \in \mathcal{O}_X \Big\}$$

$$k \geq 0 \quad V_k(D_X) = \Big\{ \sum_{i=0}^{k} \partial_t^i\, V_0(D_X) \Big\}$$

$$V_{-k}(D_X) = t^k V_0(D_X)$$

$$G_0 V(D_X) = V_0(D_X) \Big/ \underbrace{V_{-1}(D_X)}_{t V_0(D_X)} = \mathcal{D}_Z[E]$$

class of $t\partial_t$ in $Gr_0 V(D_X)$.

<u>Goal</u>: Define nearby cycles functor for D-modules as the assoc. graded to a suitable good V-filtration on M.

<u>Definition</u>: A good V-filtration on M is an exhaustive filtration $U_\bullet$ of M such that:

a) $\forall k, \ell \in \mathbb{Z}$: $V_k(D_X) \cdot U_\ell \subseteq U_{k+\ell}$

b) $\forall k \in \mathbb{Z}$: U_k is coherent $V_0(D_X)$-module

c) $\exists k_0 \in \mathbb{Z}$ $\forall k \geq 0$: $V_k(D_X)\, U_{k_0} = U_{k+k_0}$ and $V_{-k}(D_X) \cdot U_{-k_0} = U_{-k-k_0}$

Exercise: If $U_\bullet$ and $U'_\bullet$ are good V-filtrations, show that $\exists\, k_1, k_2 \in \mathbb{Z}$ such that $\forall k \in \mathbb{Z}$: $U_{k+k_1} \subset U'_k \subseteq U_{k+k_2}$

In particular, if one good V-filtration is constant, then all good V-filtrations are constant.

Theorem 1: M holonomic D_X-module

(1) M admits a good V-filtration.

(2) $\forall U_\bullet$ good V-filtration $\exists\, b \in \mathbb{C}[s]\setminus\{0\}$ such that

$b(t\partial_t + k)\, U_k \subseteq U_{k-1}$ "Bernstein-polynomial of the good V-filtration"

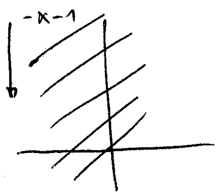
(3) $\forall \sigma$ section of $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$ $\exists!$ good V-filtration $V^\sigma(M)$ whose Bernstein-polynomial has roots in $\mathrm{Im}(\sigma)$.

Example: M holonomic, $m \in M$, $D_X \cdot m$, $(V_k(D_X)\cdot m)_{k\in\mathbb{Z}}$ is a good V-filtration

→ its Bernstein polynomial is denoted by b_m

Example: The canonical V-filtration on M. $\mathbb{C} = \mathbb{R} + i\mathbb{R}$ with lex ordering.

$\alpha \in \mathbb{C}$, $V_\alpha(M) = \{ m \in M \mid \forall x \in Z(b_m) : x \geq -\alpha - 1 \}$



Fact 3: $(V_{\alpha+k}(M))_{k\in\mathbb{Z}}$ is a good V-filtration

$\phi_t M := V_0(M)/V_{-1}(M) \in D_Z[E]$, E is the induced action of $t\partial_t$

$b :=$ Bernstein polynomial of $(V_k(M))_{k\in\mathbb{Z}}$ has roots $Z(b) \subseteq [-1,0)$, $b(E) = 0 \circlearrowright \phi_t M$ ⇝ The eigenvalues of $E \circlearrowright \phi_t M$ are in $[-1,0)$

$\alpha \in \mathbb{C}$, $V_{<\alpha}(M) := \{ m \in M \mid \forall x \in Z(b_m) : x > -\alpha - 1 \}$

$\mathrm{Gr}\, V_\alpha(M) := V_\alpha(M)/V_{<\alpha}(M)$, $\phi_t M : \bigoplus_{\alpha\in(-1,0]} \mathrm{Gr}\, V_\alpha(M)$ ↳ generalized eigenspace for $-\alpha-1$

$\psi_t M := V_{<0}(M)/V_{<-1}(M)$ $\bigoplus_{\alpha\in[-1,0)} \mathrm{Gr}\, V_\alpha(M)$

$\psi_t M \to \phi_t M$

$m \mapsto m$, $m \in \mathrm{Gr}\, V_\alpha(M)$ $\alpha \in (-1,0)$

$m \mapsto \partial_t m$ $m \in \mathrm{Gr}\, V_{-1}(M)$

Thm 2: $\mathcal{M}$ regular holonomic.

$$\begin{tikzcd} \mathrm{DR}\, i^! \mathcal{M} \arrow[r] \arrow[d, "\wr"] & \mathrm{DR}\, \Psi_f \mathcal{M} \arrow[r] \arrow[d, "\wr"] & \mathrm{DR}\, \Phi_f \mathcal{M} \arrow[d, "\wr"] & \text{action of } \exp(-2\pi i E) \arrow[d, leftrightarrow] \\ i^! \mathrm{DR}\, \mathcal{M} \arrow[r] & \Psi_f \mathrm{DR}\, \mathcal{M} \arrow[r] & \Phi_f \mathrm{DR}\, \mathcal{M} & \text{monodromy action} \end{tikzcd}$$

Example: $f : \mathbb{C}^n \to \mathbb{C}$

$$(x_1, \dots, x_n) \mapsto x_1^{a_1} \cdots x_n^{a_n}, \quad a_i \geq 1$$

[really should replace everything by a graph embedding here]

Monodromy action on $\Psi_f \mathbb{C}$? $\mathbb{C} \underset{qis}{\simeq} \mathrm{DR}(\mathcal{O}_X) \rightsquigarrow$ understand Bernstein polynomial of the good V-filtration $(V_{<k}(\mathcal{O}_X))_{k \in \mathbb{Z}} \rightsquigarrow$ Computation gives $\prod_{i=0}^{n} \prod_{j=0}^{a_i - 1} \left(s + \frac{j}{a_i}\right) \rightsquigarrow$ eigenvalues of monodromy are roots of unity (local monodromy theorem)