This is the program for the two-day workshop on Faltings's proof of Lang's conjecture organized by Darmstadt-Frankfurt-Mainz.

Overview of program

We prove (in as much detail as possible) the arithmetic part of Lang's conjecture for closed subvarieties of abelian varieties, as proven by Faltings in 1994. In other words, we prove that a closed subvariety X of an abelian variety A over a number field k which does not contain any translates of a positive-dimensional abelian subvariety of A has only finitely many rational points.

Faltings's theorem implies Mordell's conjecture, i.e., a smooth projective curve of genus at least two over k has only finitely many rational points. This theorem was proven earlier by Faltings in 1983 using different methods (e.g., Galois representations, p-divisible groups, Faltings's height function, and isogenies of abelian varieties).

Faltings's proof of his finiteness result for closed subvarieties of abelian varieties is motivated by Vojta's (different) proof of the Mordell conjecture iin 1991. Vojta's proof uses ideas from Diophantine geometry (e.g., Dyson's lemma, index of a section of a line bundle, small sections, arithmetic Riemann-Roch). Faltings simplifies certain parts of Vojta's argument (e.g., he avoids the arithmetic Riemann-Roch theorem) and obtains a more general finiteness result.

The "main" invention of Faltings is the (arithmetic and geometric) product theorem. Part of the aim of this seminar is to understand Faltings's product theorem(s).

On the first day of the seminar we will explain the outline of Faltings's proof. On the second day we will present the proofs of the main results necessary to complete the argument.

We will follow (very) closely the lecture notes of Edixhoven-Evertse on Faltings's proof.

Date

We meet on July 17th and July 18th.

- July 17th. 14:00h - 18:00h
- July 18th. 10:00h - 17:00h.

Schedule

The schedule is as follows.

Talk 1: July 17 14:00h-14:30h.

Talk 2: July 17 15:00h-15:30h.

Talk 3: July 17 16:30h-18:00h.

Talk 4: July 18 10:00h - 11:30h

Talk 5: July 18 13:00h-14:30h

Talk 6: July 18 15:30h-17:00h

References

We follow closely (and mostly) Edixhoven-Evertse's lectures *Diophantine approximation on ableian varieties*. Other references are:

1. Faltings, Ann. of Math. Diophantine Approximation on abelian varieties.

2. Vojta. Lecture notes. Applications of Arithmetic Algebraic Geometry to Diophantine Approximations https://math.berkeley.edu/ vojta/ftp/cime.pdf This is our main reference.

3. Vojta. Siegel's theorem in the compact case. Annals of Mathematics.

4. Nakamaye . Journal de Theorie de Nombres de Bordeaux. Diophantine Approximation on algebraic varieties. This paper contains the structure of the proof.

Day One, Talk 1: Overview of Lang's conjecture for projective varieties

30 mins Ariyan Javanpeykar

We give an overview of Lang's conjecture for projective varieties: a projective variety admits no non-constant morphisms from an abelian variety if and only if it is "arithmetically hyperbolic" if and only if it is "algebraically hyperbolic" if and only if it is "analytically hyperbolic".

We define the aforementioned different notions of hyperbolicity, and prove that every arithmetically hyperbolic (resp. algebraically hyperbolic, resp. analytically hyperbolic) variety admits no non-constant maps from an abelian variety.

If X is a closed subvariety of an abelian variety A, then X contains no positive-dimensional translates of an abelian subvariety of A if and only if X admits no non-constant maps from an abelian variety.

Assume that $X \subset A$ is a closed subvariety of an abelian variety which does not contain any translates of a positive-dimensional abelian subvariety of A. Then, the algebraic and analytic hyperbolicity of X was proven by Ueno and Kawamata, respectively. Faltings proved the arithmetic hyperbolicity of X. The latter theorem is the main topic of this seminar.

Theorem 1 (Faltings). *Let k be a number field and let A be an abelian variety over k. Let $X \subset A$ be a closed subvariety. Suppose that every morphism $B_{\overline{k}} \to X_{\overline{k}}$ with B an abelian variety is constant. Then $X(k)$ is finite.*

Day One, Talk 2: Structure of Faltings's proof of Theorem 1

30 mins, Matthias Nickel

We state and explain the three key results used by Faltings to prove his finiteness theorem (Theorem 1). There are three main ingredients of the proof.

1. One ingredient is a lower bound on the "index" of any section of a certain line bundle M, on a certain self-product of X, at a rational point. This lower bound is proven using geometric arguments and a bit of height theory.

2. The second ingredient is Faltings's product theorem.

3. Another ingredient is the existence of a "small section". That is, the existence of a section (of the aforementioned line bundle M) whose "norm" is bounded uniformly from above. In this part we need Faltings's version of Siegel's lemma.

In the talk, we will explain how these three main ingredients fit together to complete the proof of Theorem 1.

Day One, Talk 3: A geometric product theorem

90 mins Philipp Licht

We follow Chapter VIII of Edixhoven-Evertse by Marius van der Put. The results in this section are used in Talk 4. The main ingredients of this talk are differential operators on varieties over fields of characteristic zero, the index of a section of a line bundle at a point, and a bit of intersection theory (to state the product theorem).

Day Two, Talk 4: The geometric part of Faltings's proof

90 mins Ariyan Javanpeykar

A part of Faltings's proof relies on geometric arguments. In this talk, we present a proof of the following theorem.

Theorem 2. *Let $X \subset A$ be a closed subvariety of an abelian variety A over an algebraically closed field of characteristic zero which does not contain translates of a positive-dimensional abelian subvariety of A. Then the following statements hold.*

1. *There is a positive integer m such that the morphism*

$$X^m \to A^{m-1}, \quad (x_1, \ldots, x_m) \mapsto (2x_1 - x_2, 2x_2 - x_3, \ldots, 2x_n - x_{n-1})$$

   is finite. Fix such an integer m.

2. *Let $\mathcal{L}$ be a symmetric ample line bundle on A. Let $\epsilon, s_1, \ldots, s_m$ be positive rational numbers. Define $\mathcal{L}(-\epsilon, s_1, \ldots, s_m)$ to be a certain line bundle on X^m (as defined on p. 85 in Edixhoven-Evertse's book). Then, there is an $\epsilon_0 > 0$ such that, if $\epsilon < \epsilon_0$, the degree of any product subvariety $Y \subset X^m$ with respect to $\mathcal{L}(-\epsilon, s_1, \ldots, s_m)$ is positive.*

3. *If $\epsilon < \epsilon_0$, then there is an integer s such that, for any $s_1, \ldots, s_m$ with $s_i/s_{i+1} \geq s$, the (restriction of the) line bundle $\mathcal{L}(-\epsilon, s_1, \ldots, s_m)$ (to X^m) is ample on X^m.*

Our presentation follows Chapter IX of Edixhoven-Evertse. We will use certain results from Chapter VIII in Edixhoven-Evertse (proven in Talk 4).

Day Two, Talk 5: Small sections and Faltings's version of Siegel's lemma

90 mins Matthias Nickel

We explain what "small sections" are. We also explain how one uses Siegel's lemma to obtain small sections. We follow Chapter X of Edixhoven-Evertse by Robert-Jan Kooman.

Day Two, Talk 6: Arithmetic part of Faltings's proof

90 mins Speaker: Robert Wilms

This talk completes the proof of Faltings's theorem (Theorem 1) and combines all the earlier results. We follow Edixhoven's Chapter XI.