An overview of D-modules: holonomic D-modules, b-functions, and V-filtrations

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The sheaf of differential operators

Let X be a smooth complex algebraic variety, with $\dim(X) = n$.
The sheaf of differential operators $\mathcal{D}_X$ on X is the subsheaf of $\mathbb{C}$-algebras of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and $\mathcal{T}_X = \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_X)$.

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If $x_1, \ldots, x_n$ are algebraic coordinates on an open subset $U \subseteq X$, then we have derivations $\partial_1, \ldots, \partial_n$ on U such that $\partial_i(x_j) = \delta_{i,j}$, and $\mathcal{D}_X|_U$ is a locally free (left or right) $\mathcal{O}_X$-module with basis $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$, $\beta \in \mathbb{N}^n$.

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Example. If $X = \mathbb{A}^n$, then we have the Weyl algebra

$$A_n(\mathbb{C}) = \Gamma(X, \mathcal{D}_X) = \frac{\mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle}{[x_i, x_j] = 0, [\partial_i, \partial_j] = 0, [\partial_i, x_j] = \delta_{i,j}}.$$

The order filtration

$\mathcal{D}_X$ carries a filtration by order of differential operators: $F_p\mathcal{D}_X$ is the locally free submodule of $\mathcal{D}_X$ generated by those ∂^β with $|\beta| = \sum_i \beta_i \leq p$. For example, we have $F_0\mathcal{D}_X = \mathcal{O}_X$ and $F_1\mathcal{D}_X = \mathcal{O}_X + \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_X)$.

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Easy properties:

1. $F_p\mathcal{D}_X \cdot F_q\mathcal{D}_X \subseteq F_{p+q}\mathcal{D}_X$ for all $p, q \geq 0$.
2. $[F_p\mathcal{D}_X, F_q\mathcal{D}_X] \subseteq F_{p+q-1}\mathcal{D}_X$ for all $p, q \geq 0$.

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This implies that the corresponding graded object

$$\mathrm{gr}^F_\bullet(\mathcal{D}_X) := \bigoplus_{p\geq 0} F_p\mathcal{D}_X / F_{p-1}\mathcal{D}_X$$

is a sheaf of graded commutative rings. We thus have a morphism

$$\pi_*(\mathcal{O}_{T^*X}) = \mathrm{Sym}^\bullet_{\mathcal{O}_X}(\mathcal{T}_X) \to \mathrm{gr}^F_\bullet(\mathcal{D}_X)$$

and the local description shows that this is an isomorphism.

Left $\mathcal{D}_X$-modules

Giving a left $\mathcal{D}_X$-module is the same as an $\mathcal{O}_X$-module $\mathcal{M}$ with an action

$$\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_X) \times \mathcal{M} \to \mathcal{M}$$

that satisfies:

1. $D \cdot (fm) - f(D \cdot m) = D(f)m$ for all $f \in \mathcal{O}_X$.
2. $D_1 \cdot (D_2 \cdot m) - D_2 \cdot (D_1 \cdot m) = [D_1, D_2] \cdot m$.

Equivalently, this can be reformulated as a $\mathbb{C}$-linear map $\nabla\colon \mathcal{M} \to \Omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$. First condition: this is a connection, second condition: the connection is integrable.
The De Rham complex of $\mathcal{M}$, situated in degrees $-n, \ldots, 0$ is

$$0 \to \mathcal{M} \to \Omega_X \otimes_{\mathcal{O}_X} \mathcal{M} \to \Omega_X^2 \otimes_{\mathcal{O}_X} \mathcal{M} \to \ldots \to \Omega_X^n \otimes_{\mathcal{O}_X} \mathcal{M} \to 0.$$

Left $\mathcal{D}_X$-modules, cont'd

All $\mathcal{D}_X$ modules we will consider will be quasi-coherent as $\mathcal{O}_X$-modules.

Fact: Every $\mathcal{D}_X$-module which is coherent as an $\mathcal{O}_X$-module is locally free. These are the "nice objects" in the category of $\mathcal{D}_X$-modules.

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Example 1. The structure sheaf $\mathcal{O}_X$ has a tautological left $\mathcal{D}_X$-action. The corresponding connection is the De Rham differential $d\colon \mathcal{O}_X \to \Omega_X$.

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Example 3. If Z is a hypersurface in X, then

$$\mathcal{O}_X(*Z) = \bigcup_{m\geq 1} \mathcal{O}_X(mZ)$$

has a left $\mathcal{D}_X$-module structure induced by the one on $\mathcal{O}_X$, via the "quotient rule". If $Z = (f = 0)$, then $\mathcal{O}_X(*Z) = \mathcal{O}_X[1/f]$. Note: this is *not* coherent as an $\mathcal{O}_X$-module. This will be a key example for us.

Filtrations on coherent $\mathcal{D}_X$-modules

A $\mathcal{D}_X$-module is coherent if locally finitely generated over $\mathcal{D}_X$. Such a module $\mathcal{M}$ is studied by choosing an (increasing, exhaustive) good filtration $F_\bullet\mathcal{M}$ on $\mathcal{M}$:

1. $F_p\mathcal{M} \subseteq \mathcal{M}$ coherent $\mathcal{O}_X$-submodule for all p, and it is 0 for $p \ll 0$.
2. $F_p\mathcal{D}_X \cdot F_q\mathcal{M} \subseteq F_{p+q}\mathcal{M}$ for all p and q, with equality for all p if $q \gg 0$ (then say the filtration is determined at level q).

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It is easy to construct such a filtration: e.g. choose a coherent $\mathcal{O}_X$-submodule $\mathcal{M}_0$ in $\mathcal{M}$ such that $\mathcal{D}_X \cdot \mathcal{M}_0 = \mathcal{M}$ and let

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Clearly: such a good filtration is far from unique.

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The definition of a good filtration implies that

$$\mathrm{gr}_\bullet^F(\mathcal{M}) := \bigoplus_{p\in\mathbb{Z}} F_p\mathcal{M}/F_{p-1}\mathcal{M}$$

is locally finitely generated over $\mathrm{gr}_\bullet^F(\mathcal{D}_X)$: it is generated in degree $\leq q$ if the filtration is determined at level q. Hence this can be considered a coherent sheaf on T^*X. Its support is the characteristic variety $\mathrm{Char}(\mathcal{M})$. Key point: this is independent of the choice of filtration.

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The dimension $\dim(\mathcal{M})$ of $\mathcal{M}$ is the dimension of $\mathrm{Char}(\mathcal{M}) \subseteq T^*X$.

Holonomic $\mathcal{D}_X$-modules

We have the following "miracle":

Theorem 1 (Bernstein, Sato-Kawai-Kashiwara). For every nonzero coherent $\mathcal{D}_X$-module $\mathcal{M}$, we have $\dim(\mathcal{M}) \geq n$.

For a sketch of proof in the case $X = \mathbb{A}^n$: see slides at the end.

Definition. A coherent $\mathcal{D}_X$-module is holonomic if $\mathcal{M} = 0$ or $\dim(\mathcal{M}) = n$.

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Example 1. If $\mathcal{M} = \mathcal{D}_X$, then $\mathrm{Char}(\mathcal{M}) = T^*X$, hence $\dim(\mathcal{D}_X) = 2n$.

Example 2. If $\mathcal{E}$ is a vector bundle with integrable connection, we can take $F_p\mathcal{E} = 0$ for $p < 0$ and $F_p\mathcal{E} = \mathcal{E}$ for $p \geq 0$, hence $\mathrm{Char}(\mathcal{E})$ is the 0-section of T^*X. Therefore $\mathcal{E}$ is a holonomic $\mathcal{D}_X$-module.

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Example 3. If Z is a hypersurface in X, then $\mathcal{O}_X(*Z)$ is holonomic (note: even coherence is not clear). For a sketch of proof when $X = \mathbb{A}^n$, see slides at the end.

Holonomic $\mathcal{D}_X$-modules, cont'd

For a short exact sequence of coherent $\mathcal{D}_X$-modules

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0,$$

we choose a good filtration on $\mathcal{M}$ and then the induced filtrations on $\mathcal{M}'$ and $\mathcal{M}''$. We get a short exact sequence

$$0 \to \mathrm{gr}_\bullet^F(\mathcal{M}') \to \mathrm{gr}_\bullet^F(\mathcal{M}) \to \mathrm{gr}_\bullet^F(\mathcal{M}'') \to 0,$$

and thus $\mathrm{Char}(\mathcal{M}) = \mathrm{Char}(\mathcal{M}') \cup \mathrm{Char}(\mathcal{M}'')$.
Theorem 1 implies that $\mathcal{M}$ is holonomic iff both $\mathcal{M}'$ and $\mathcal{M}''$ are holonomic. In particular, holonomic $\mathcal{D}_X$-modules form an Abelian subcategory of all $\mathcal{D}_X$-modules.

Holonomic $\mathcal{D}_X$-modules, cont'd

Using the graded structure, one defines a notion of multiplicity $e(\mathcal{M})$ for a coherent $\mathcal{D}_X$-module $\mathcal{M}$ (analogue of degree of coherent sheaf on $\mathbf{P}^n$). This is a non-negative integer and it is 0 if and only if $\mathcal{M} = 0$.

For a version of the definition for $X = \mathbb{A}^n$: see the last slides.

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For an exact sequence of holonomic $\mathcal{D}_X$-modules as on the previous slide, we have

$$e(\mathcal{M}) = e(\mathcal{M}') + e(\mathcal{M}'').$$

Consequence: all objects in the category of holonomic $\mathcal{D}_X$-modules have finite length.

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Another reason holonomic $\mathcal{D}_X$-modules are good: they are preserved by pull-back and push-forward of $\mathcal{D}_X$-modules (not true for coherent $\mathcal{D}_X$-modules).

Left vs. right $\mathcal{D}_X$-modules

There is a canonical equivalence of categories between left $\mathcal{D}_X$-modules and right $\mathcal{D}_X$-modules. This is useful, in practice, since the push-forward functor is naturally defined for right $\mathcal{D}$-modules.

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Key point: The sheaf $\omega_X = \Omega_X^n$ has a natural structure of right $\mathcal{D}_X$-module given for an n-form ω and a derivation D by

$$\omega \cdot D = \mathrm{Lie}(D)\omega, \quad \text{that maps}$$

$$(D_1, \ldots, D_n) \quad \text{to} \quad D\big(\omega(D_1, \ldots, D_n)\big) - \sum_{i=1}^{n} \omega\big(D_1, \ldots, [D, D_i], \ldots, D_n\big).$$

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If $x_1, \ldots, x_n$ are local coordinates, then

$$(f dx_1 \wedge \ldots \wedge dx_n)\partial_i = -\frac{\partial f}{\partial x_i} dx_1 \wedge \ldots \wedge dx_n.$$

Left vs. right $\mathcal{D}_X$-modules, cont'd

If $\mathcal{M}$ is a left $\mathcal{D}_X$-module, then $\omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is a right $\mathcal{D}_X$-module, with

$$(\omega \otimes m)D = \omega D \otimes m - \omega \otimes Dm.$$

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If $\mathcal{N}$ is a right $\mathcal{D}_X$-module, then $\mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{N})$ is a left $\mathcal{D}_X$-module, with

$$(D\varphi)(\omega) = \varphi(\omega D) - \varphi(\omega)D.$$

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The canonical isomorphisms of $\mathcal{O}_X$-modules

$$\mathcal{M} \to \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \omega_X \otimes_{\mathcal{O}_X} \mathcal{M}), \quad \omega_X \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{N}) \to \mathcal{N}$$

are morphisms of $\mathcal{D}_X$-modules, hence we get our equivalence between the categories of left and right $\mathcal{D}_X$-modules.

Left vs. right $\mathcal{D}_X$-modules, cont'd

Description in local coordinates $x_1, \ldots, x_n$ on U: we have a $\mathbb{C}$-linear anti-commutative map $\tau\colon \mathcal{D}_X \to \mathcal{D}_X$ such that

$$\tau|_{\mathcal{O}_X} = \mathrm{id}_{\mathcal{O}_X} \quad \text{and} \quad \tau(\partial_i) = -\partial_i.$$

Note that the local coordinates induce a trivialization $\omega_X|_U \simeq \mathcal{O}_U$.
A left $\mathcal{D}_X$-module M becomes a right $\mathcal{D}_X$-module via τ and conversely.
This is the local description of the previous equivalence.

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It is clear that from a good filtration $F_\bullet\mathcal{M}$ on the left $\mathcal{D}_X$-module $\mathcal{M}$, we obtain a filtration on the corresponding right $\mathcal{D}_X$-module $\mathcal{N} = \omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$.
Indexing convention: $F_{p-n}\mathcal{N} = \omega_X \otimes_{\mathcal{O}_X} F_p\mathcal{M}$.

The Riemann-Hilbert correspondence

Classical version. We work in the analytic topology, with holomorphic vector bundles. Given a rank r holomorphic vector bundle, with integrable connection $(\mathcal{E}, \nabla)$, the classical result about solving ODE implies that $\ker(\nabla) \subseteq \mathcal{E}$ is a rank r local system.

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Conversely, given a local system L, we get a vector bundle $\mathcal{E} = L \otimes_{\mathbb{C}} \mathcal{O}_X$, with integrable connection $1_L \otimes d$. We thus get an equivalence of categories between the category of holomorphic vector bundles on X, with integrable connection, and the category of local systems on X.

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If X is compact, then GAGA implies that algebraic vector bundles with integrable connection are the same as holomorphic vector bundles with integrable connection.

The Riemann-Hilbert correspondence, cont'd

Non-compact case: version due to Deligne describing local systems via algebraic vector bundles with integrable connection, and having regular singularities. For curves, this means that the connection satisfies the Fuchs condition at the points in the boundary of X.

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Theorem 2 (Kashiwara, Mebkhout, Beilinson-Bernstein) For every smooth algebraic variety X, the De Rham functor DR_X extends to an equivalence of triangulated categories

$$D^b_{rh}(\mathcal{D}_X) \simeq D^b_{\mathrm{constr}}(X), \quad \text{where}$$

$D^b_{rh}(\mathcal{D}_X)$: bounded derived category of complexes of $\mathcal{D}_X$-modules, with regular holonomic cohomology, and
$D^b_{\mathrm{constr}}(X)$: the bounded derived category of complexes with constructible cohomology.

The Riemann-Hilbert correspondence, cont'd

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The Riemann-Hilbert correspondence, cont'd

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The correspondence is compatible with the functors (push-forward, pull-back, etc) defined on both sides.

The Bernstein-Sato polynomial

The following result was a key motivation in the development of the algebraic theory of $\mathcal{D}$-modules.

Theorem 3 (Bernstein). If X is a smooth algebraic variety and $f \in \mathcal{O}(X)$ nonzero, then there is a nonzero $b(s) \in \mathbb{C}[s]$ such that

$$b(s)f^s = P(s) \cdot f^{s+1}, \quad \text{for some} \quad P(s) \in \mathcal{D}_X[s].$$

Here we consider the left $\mathcal{D}_X$-module $\mathcal{O}_X[s, 1/f]f^s$, where

$$D \cdot f^s = \frac{sD(f)}{f} f^s \quad \text{for every} \quad D \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_X).$$

The set of those $b(s)$ for which there is $P(s)$ as above is an ideal. The monic generator of this ideal is the Bernstein-Sato polynomial $b_f(s)$ of f.

The Bernstein-Sato polynomial, cont'd

Example 1. If $f = x_1 \in \mathbb{C}[x_1, \ldots, x_n]$, then

$$(s+1)f^s = \partial_1 \cdot f^{s+1}; \quad \text{in fact} \quad b_f(s) = s+1.$$

The Bernstein-Sato polynomial, cont'd

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Example 2. If $f = x_1^2 + \ldots + x_n^2 \in \mathbb{C}[x_1, \ldots, x_n]$, then

$$(s+1)(4s+2n)f^s = (\partial_1^2 + \ldots + \partial_n^2) \cdot f^{s+1}.$$

In fact, we have $b_f(s) = (s+1)(s+\frac{n}{2})$.

The Bernstein-Sato polynomial, cont'd

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In fact, we have $b_f(s) = (s+1)(s+\frac{n}{2})$.

Example 3. For every f, by making $s = -1$, we have

$$b_f(-1)\frac{1}{f} = P(-1) \cdot 1 \in \mathcal{O}(X).$$

Hence if f is non-invertible, then $(s+1)$ divides $b_f(s)$.

The Bernstein-Sato polynomial, cont'd

Example 4. For every positive integer m, we have

$$b_f(-m-1)\frac{1}{f^{m+1}} \in \mathcal{D}_X \cdot \frac{1}{f^m}$$

Hence $\frac{1}{f^m}$ generates $\mathcal{O}_X[1/f]$ over $\mathcal{D}_X$ if b_f has no integer roots $< -m$.

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Theorem 4 (Kashiwara). All roots of $b_f(s)$ are in $\mathbb{Q}_{<0}$.

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Theorem 4 (Kashiwara). All roots of $b_f(s)$ are in $\mathbb{Q}_{<0}$.

In general, the roots of $b_f(s)$ are subtle invariants of the singularities of f, related to several other such invariants. For example, the largest root of $b_f(s)$ is $-\mathrm{lct}(f)$, by a result of Lichtin and Kollár.

The Bernstein-Sato polynomial, cont'd

Sketch of proof for the existence of the Bernstein-Sato polynomial:

Suppose, for simplicity, $X = \mathbb{A}^n$. Consider $M = \mathbb{C}(s)[x_1, \ldots, x_n]_f \cdot f^s$ as a module over $A_n\big(\mathbb{C}(s)\big)$.

M is a holonomic $\mathcal{D}$-module (this is proved in the same way that one shows that $\mathbb{C}[x_1, \ldots, x_n]_f$ is holonomic over $A_n(\mathbb{C})$). Therefore the following decreasing sequence of submodules is stationary

$$A_n\big(\mathbb{C}(s)\big) \cdot f^s \supseteq A_n\big(\mathbb{C}(s)\big) \cdot f^{s+1} \supseteq A_n\big(\mathbb{C}(s)\big) \cdot f^{s+2} \supseteq \ldots$$

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We obtain a relation $f^{m+s} = P(s, x, \partial_x) \cdot f^{m+s+1}$. The theorem follows using the automorphism $s \to s - m$ and clearing the denominators.

The V-filtration

The V-filtration was introduced by Malgrange (and extended by Kashiwara) in order to describe the nearby cycles functor at the level of holonomic $\mathcal{D}$-modules.

Setting: Let $f \in \mathcal{O}(X)$ nonzero. Consider

$$\iota\colon X \hookrightarrow X \times \mathbb{A}^1, \quad \iota(x) = \big(x, f(x)\big)$$

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If $\mathcal{M}$ is a $\mathcal{D}_X$-module, then the $\mathcal{D}$-module theoretic push-forward $\iota_+ M$ is described as follows:

Case 1. If $\mathcal{M} = \mathcal{O}_X$, then

$$\iota_+\mathcal{M} = \mathcal{O}_X[t]_{f-t}/\mathcal{O}_X[t] = \bigoplus_{j\geq 0} \mathcal{O}_X \cdot \partial_t^j \delta, \quad \text{where} \quad \delta = [1/(f-t)].$$

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Case 2. For any $\mathcal{M}$, we have $\iota_+\mathcal{M} = \mathcal{M} \otimes_{\mathcal{O}_X} \iota_+\mathcal{O}_X$.

The V-filtration, cont'd

Goal of V-filtration: put action of $\partial_t t$ on $\iota_+\mathcal{M}$ in upper-triangular form.

The V-filtration, cont'd

Goal of V-filtration: put action of $\partial_t t$ on $\iota_+\mathcal{M}$ in upper-triangular form. It is a decreasing, exhaustive, rational, discrete, left-continuous filtration $(V^\alpha = V^\alpha \iota_+\mathcal{M})_{\alpha\in\mathbf{Q}}$ on $\iota_+\mathcal{M}$ that satisfies:

i) Each V^α is finitely generated over $\mathcal{D}_X[t, \partial_t t]$

ii) $t \cdot V^\alpha \subseteq V^{\alpha+1}$, with equality if $\alpha > 0$

iii) $\partial_t \cdot V^\alpha \subseteq V^{\alpha-1}$

iv) $\partial_t t + \alpha$ is nilpotent on $V^\alpha / V^{>\alpha}$.

These properties uniquely characterize the V-filtration.

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- iv) $\partial_t t + \alpha$ is nilpotent on $V^\alpha/V^{>\alpha}$.

These properties uniquely characterize the V-filtration.
Key observation: if $\mathcal{M} = \mathcal{O}_X$, then

$$\iota_+\mathcal{O}_X[1/f] \simeq \mathcal{O}_X[1/f, s]f^s, \delta \to f^s$$

where s acts on the left-hand side by $-\partial_t t$ and t acts on the right-hand side by $P(s)f^s \to P(s+1)f^{s+1}$.

The V-filtration, cont'd

We thus see that the Bernstein-Sato polynomial b_f is the monic polynomial of smallest degree such that

$$b_f(-\partial_t t)\delta \in \mathcal{D}_X[-\partial_t t] \cdot t\delta$$

The V-filtration, cont'd

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Using this, and the rationality of the roots of b_f, Malgrange showed the existence of V-filtration for $\mathcal{M} = \mathcal{O}_X$. He also showed that $\mathrm{DR}_X(V^{>0}/V^{>1})[1]$ gives the nearby cycles of f, with the monodromy action corresponding to the action of $\exp(-2\pi i\partial_t t)$.

Consequence: the eigenvalues of the monodromy action on the cohomology of the Milnor fiber of f are the $\exp(2\pi i\alpha)$, where α is a root of b_f.

Appendix: some proofs in the case $X = \mathbb{A}^n$

When $X = \mathbb{A}^n$, some of the basic results are easier to prove. The reason: we can consider on $A_n = A_n(k)$ (where k is any field of characteristic 0) the Bernstein filtration:

$$B_p A_n = \bigoplus_{|\alpha|+\beta|\leq n} kx^{\alpha}\partial^{\beta}.$$

As before, we have

$$\mathrm{gr}_{\bullet}^{B} A_n \simeq S := \mathrm{Sym}_{R}^{\bullet}\mathrm{Der}_k(R) \simeq k[x_1, \ldots, x_n, y_1, \ldots, y_n],$$

where $R = k[x_1, \ldots, x_n]$.

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where $R = k[x_1, \ldots, x_n]$.

If M is a finitely generated module over A_n, we consider a filtration $F_{\bullet} M$ on M that is compatible with the Bernstein filtration on A_n (and is good, as before). We get a notion of dimension as before and a notion of multiplicity (the fact that the two notions of dimension agree can be proved using a homological characterization of dimension).

Appendix: some proofs in the case $X = \mathbb{A}^n$, cont'd

Given M, choose $F_\bullet M$ and put

$$\dim(M) := \dim\left(\mathrm{gr}^F_\bullet(M)\right) \quad \text{and} \quad e(M) := \deg\left(\mathrm{gr}^F_\bullet(M)\right).$$

Hence

$$\dim(M) = r \quad \text{iff} \quad \dim_k F_pM \sim p^r \quad \text{and}$$

$$e(M) = \lim_{p\to\infty} \frac{r! \cdot \dim_k F_pM}{p^r}, \quad \text{where} \quad r = \dim(M).$$

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Sketch of proof of Bernstein's dimension inequality:
May assume $F_0M \neq 0$. One shows by induction on p that the map

$$B_pA_n \to \mathrm{Hom}_k(F_pM, F_{2p}M), \quad Q \to (m \to Qm)$$

is injective. Since $\dim_k B_pA_n$ grows like p^{2n} and both $\dim_k F_pM$ and $\dim_k F_{2p}M$ grow like $p^{\dim(M)}$, we get $\dim(M) \geq n$.

Appendix: some proofs in the case $X = \mathbb{A}^n$, cont'd

Sketch of proof for the fact that R_f is holonomic:
Let $d = \deg(f)$. Consider on R_f the filtration given by

$$F_pR_f = \left\{ \frac{g}{f^p} \mid \deg(g) \leq p(d+1) \right\}.$$

One checks: if $\epsilon > 0$, then

$$\dim_k F_pR_f \leq (1+\epsilon)\frac{(d+1)^n}{n!}p^n \quad \text{for} \quad p \gg 0.$$

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The Bernstein inequality implies that every finitely generated submodule of R_f is holonomic, of multiplicity $\leq (d+1)^n$. Hence every increasing sequence of finitely generated submodules of R_f has length $\leq (d+1)^n$. Therefore R_f is finitely generated, and thus holonomic.