# Cohomology and algebraic cycles on real varieties.

(based on joint work with O. Benoist)

- real alg. geom ; more specifically coh. / alg. cycles

- $\mathbb{R}$ : intermediate between $\mathbb{C}$ and $\mathbb{Q}$

- some arith. flavour but OTOH, close to $\mathbb{C}$ $\rightsquigarrow$ Hodge theory can play a rôle, will emphasize

F field. Is $-1$ a sum of squares in F ? If so, how many squares are needed ?

**Exercise.** $F = \mathbb{Q}_p$. Answer: yes ; 1 if $p \equiv 1 \bmod 4$, 2 if $p \equiv 3 \bmod 4$, 4 if $p = 2$.

What if $F = \mathbb{R}(X)$, X an irred. variety / $\mathbb{R}$ ?

1st question understood for a long time ; 2nd one is very much open, turns out to have connections with cohomology, geometry, topology, Hodge theory of $X_{\mathbb{C}}$ ...

## 1) Artin's and Pfister's theorems.

X smooth, proper, irreducible / $\mathbb{R}$.

**Theorem 1.1.** (E. Artin, 1927). $-1$ is a sum of squares in $\mathbb{R}(X) \Longleftrightarrow X(\mathbb{R}) = \emptyset$.

**Example.** $\Gamma \subset \mathbb{P}^2_{\mathbb{R}}$ : $x^2 + y^2 + z^2 = 0$.

**Proof of $\Rightarrow$).** $-1 = \Sigma f_i^2$. $U \subset X$ open where the $f_i$'s are defined.

Then $U(\mathbb{R}) = \emptyset$. $(-1 = \Sigma f_i(u)^2)$.

$U(\mathbb{R}) = \emptyset \Longrightarrow X(\mathbb{R}) = \emptyset$ as X is smooth (inverse function theorem for $\begin{matrix} V \xrightarrow{\text{étale}} \mathbb{A}^d_{\mathbb{R}} \\ \cap \\ X \end{matrix}$ ) ( $X(\mathbb{R})$ locally like $\mathbb{R}^d$ )

For $\Leftarrow$) need preliminaries.

**Def.** An *ordered* field is a field F with a total order $\leq$ such that

$a \leq b \Rightarrow a + c \leq b + c$ $(\forall a, b, c \in F)$.

$\left.\begin{matrix} 0 \leq a \\ 0 \leq b \end{matrix}\right) \Rightarrow 0 \leq ab$

**Ex:** $\mathbb{R}$, (subfields of $\mathbb{R}$,) $\mathbb{R}((t))$ with $0 < t < \alpha$ for any $\alpha \in \mathbb{R}_{>0}$ "t infinitely small, $\frac{1}{t}$ infinitely large" (non-archimedean)

**Prop. 1.2.** (Artin–Schreier, 1926) A field F can be ordered iff $-1$ is not a sum of squares in F.

**Proof** (Serre 1949 (first paper)). Suppose $-1$ is not a sum of squares.

**Def.** A *preordering* on F is a subset $T \subseteq F$ s.t. $T + T \subseteq T$, $T . T \subseteq T$, $F^2 \subseteq T$.

$-1 \notin T$,

**Example:** $T = \Sigma F^2$.

Zorn: $\exists$ maximal preordering T. Then $F = T \cup (-T)$ (if $x, -x \notin T$ then $T + xT$ is a preordering)

$x \leq y \underset{\text{def}}{\Longleftrightarrow} y - x \in T$

**Prop. 1.3.** $\forall$ F ordered field containing $\mathbb{R}$, $\exists$ canonical valuation on F with residue field $\mathbb{R}$. (ex: $\mathbb{R}((t))$)

**Proof:** $\mathcal{O}_F := \{ x \in F \,;\, \exists\, n \in \mathbb{N},\ |x| \leq n \}$ ("not so large")

$m_F := \{ x \in F \,;\, \forall\, n \in \mathbb{N}_{>0},\ |x| \leq \frac{1}{n} \}$ ("so small")

$\mathrm{Frac}\, \mathcal{O}_F = F$, $\mathcal{O}_F / m_F = \mathbb{R}$.

**Exercise:** $\mathcal{O}_F$ is a valuation ring,

**Proof of Th. 1.1 $\Leftarrow$).** Suppose $-1$ is not a sum of squares in $\mathbb{R}(X)$. Then $F = \mathbb{R}(X)$ can be ordered.

$$\begin{tikzcd} & \mathrm{Spec}\,\mathbb{R}(X) \arrow[r] \arrow[d] & X \arrow[d] & \text{properness} \\ \mathrm{Spec}\,\mathbb{R} \arrow[r] \arrow[urr] & \mathrm{Spec}\,\mathcal{O}_F \arrow[r] \arrow[ur, "\exists"] & \mathrm{Spec}\,\mathbb{R} & \end{tikzcd}$$

so $X(\mathbb{R}) \neq \emptyset$.

**Def.** The *level* of a field F is $s(F) := \min\{n \, ; \, -1 \text{ sum of } n \text{ squares in } F\}$ ($\min \phi = \infty$).

**Theorem 1.4** (Pfister 1965/1967) [recall $\mathbb{Q}_p$ : 1, 2, 4]

(i) $X/\mathbb{R}$ irred., $d = \dim X$, $X(\mathbb{R}) = \phi$. Then $s(\mathbb{R}(X)) \leq 2^d$.
(ii) $\forall$ field F, $s(F)$ is a power of 2 (or $\infty$).

Proof rests on
- the theory of quadratic forms (Pfister forms)
- the theorem of Tsen-Lang ("$\mathbb{C}(X)$ is $C_d$").

the ~~bound~~ $2^d$ comes from $C_d$ for quadrics (point = $C_d \Rightarrow$ every quadratic form of dim $2^{\dim X}+1$ is isotropic). nothing to do with geometry; quad. form theory; see books by Kahn, Lam

Def: a field F is $C_i$ if ... $H \subset \mathbb{P}^N$

Conj 1.5 (Lang, 1952). $X/\mathbb{R}$ irred. dim. $d$, $X(\mathbb{R}) = \phi$. $\mathbb{R}(X)$ is $C_d$.

Rk: 1.5 $\Rightarrow$ 1.4 (i) $H : x_0^2 + \dots + x_{2^d}^2 =$

. Tsen / open.

## 2) What is the level of $\mathbb{R}(X)$?

$X/\mathbb{R}$ smooth, proper, geom. irreducible, $X(\mathbb{R}) = \phi$.

$d = 1$. $s(\mathbb{R}(X)) = 2$.

$d = 2$. ——— $\in \{2, 4\}$. 2 occurs: $X = C \times \mathbb{P}^1_{\mathbb{R}}$.

[Can 4 occur?] [yes; first appearance of Hodge theory]

**Prop. 2.1.** (Colliot-Thélène, 1993) (K3 surface)
Let $X \subset \mathbb{P}^3_{\mathbb{R}}$ be a very general quartic surface with $X(\mathbb{R}) = \phi$.
Then $s(\mathbb{R}(X)) = 4$.

$d = 3$. $s(\mathbb{R}(X)) \in \{2, 4, 8\}$.

Open question: does there exist a real threefold X such that $s(\mathbb{R}(X)) = 8$ ??

Proof of Prop. 2.1 (sketch). ① Suppose $s(\mathbb{R}(X)) < 4$. Then $s(\mathbb{R}(X)) = 2$, i.e. $-1$ is a sum of 2 squares in $\mathbb{R}(X)$, i.e. the conic $-x^2 - y^2 = z^2$ has a rat'l point over $\mathbb{R}(X)$.

i.e. $\mathrm{Br}\,\mathbb{R} \longrightarrow \mathrm{Br}\,\mathbb{R}(X)$ is the 0 map.

$\mathrm{Br}\,\mathbb{R} = \mathbb{Z}/2\mathbb{Z} \cdot (-1,-1)$

Recall: $a, b \in F^*$. $ax^2 + by^2 = z^2$ has an F-point iff $\underbrace{(a,b)}_{\text{class of quaternion algebra}} \in \mathrm{Br}\,F$ vanishes. (char $F \neq 2$)

② Noether-Lefschetz: $\mathrm{Pic}\, X_{\mathbb{C}} = \mathbb{Z} \cdot \mathcal{O}(1)$ for X very general.

③ Lemma: $\exists$ exact sequence $0 \to \mathrm{Pic}\, X \longrightarrow (\mathrm{Pic}\, X_{\mathbb{C}})^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \longrightarrow \mathrm{Br}\,\mathbb{R} \longrightarrow \mathrm{Br}\,\mathbb{R}(X$

[true for any smooth proper geom. irred. X / any field]

Proof: exercise: apply $H^0(G, -)$ to
$$0 \longrightarrow \mathbb{C}(X)^*/\mathbb{C}^* \longrightarrow \mathrm{Div}(X_{\mathbb{C}}) \longrightarrow \mathrm{Pic}(X_{\mathbb{C}}) \longrightarrow 0$$
and note $H^1(G, \mathrm{Div}\, X_{\mathbb{C}}) = 0$ (Shapiro)

use Hilbert 90 ($H^1(G, \mathbb{C}^*) = H^1(G, \mathbb{C}(X)^*) = 0$)
and $H^2(G, \mathbb{C}^*) = \mathrm{Br}\,\mathbb{R}$, $H^2(G, \mathbb{C}(X)^*) \hookrightarrow \mathrm{Br}(\mathbb{R}(X))$.

OR: $0 \to \mathbb{C}^* \longrightarrow \mathbb{C}(X)^* \longrightarrow \mathbb{C}(X)^*/\mathbb{C}^* \longrightarrow 0 \rightsquigarrow H^0(G, \mathbb{C}(X)^*/\mathbb{C}^*) = \mathbb{R}(X)^*/\mathbb{R}^*$
$H^1(G, \text{———}) = \mathrm{Ker}(\mathrm{Br}\,\mathbb{R} \to \mathrm{Br}\,\mathbb{R}($

Noether-Lefschetz from Hodge theory:
$$\mathrm{Pic}\, X_{\mathbb{C}} = \underbrace{H^2(X(\mathbb{C}), \mathbb{Z})}_{\text{fixed}} \cap \underbrace{H^{1,1}}_{\text{varies with } X} \subset \underbrace{H^2(X(\mathbb{C}), \mathbb{C})}_{\text{fixed}}.$$

Point: $H^2(X, \mathcal{O}_X) \neq$
so $H^{1,1} \neq H^2(X(\mathbb{C}),$

Note: Some quartics satisfy $s(\mathbb{R}(X)) = 2$, e.g. $x^4 + y^4 + z^4 + w^4 = 0$ (clearly $-1$ is a sum of 3 squares, hence of 2).

What if $H^2(X, \mathcal{O}_X) = 0$?

Theorem 2.2. (B-W, 2017) $X$ smooth proper geom. irred. surface $/\mathbb{R}$, with $X(\mathbb{R}) = \phi$. If $H^2(X, \mathcal{O}_X) = 0$, then $s(\mathbb{R}(X)) = 2$, i.e. $-1$ is a sum of 2 squares in $\mathbb{R}(X)$.

Previously known for $X_{\mathbb{C}}$ rational or Enriques (Parimala-Sujatha 1991, Sujatha-van Hamel 2000)
[relies on classification of $X(\mathbb{R})$, $X$ Enriques by Degtyarev, Kharlamov + results on Br(Enriques) (Nikulin-Sujatha, Mangolte, vH)].

END OF FIRST LECTURE

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Recall: criterion for $-1 \in \Sigma F^2$, $F = \mathbb{R}(X)$; defined level and explained how for surfaces it is connected to the cohomology / Hodge theory of the surface. State Theorem. Explain proof + tools → will shift attention geometry to other problems, looking for curves IHC...

## 3) Equivariant cohomology.

$G = \mathrm{Gal}(\mathbb{C}|\mathbb{R}) = \{1, \sigma\}$. $V$ topological space with a continuous action of $G$.

Def. • A G-equivariant sheaf $\mathcal{F}$ on $V$ is a sheaf $\mathcal{F}$ endowed with an iso. $u : \mathcal{F} \to \sigma^*\mathcal{F}$ s.t. $\mathcal{F} \xrightarrow{u} \sigma^*\mathcal{F} \xrightarrow{\sigma^* u} \sigma^*\sigma^*\mathcal{F} = \mathcal{F}$ (composite $\mathrm{Id}$).

Example: constant sheaf associated to a $G$-module.

- $H^p_G(V, -) :=$ $p$-th derived functor of

$$\begin{tikzcd}(G\text{-eq. sheaves of ab. gp. on } V) \arrow[r] & \mathrm{Ab} \\ \mathcal{F} \arrow[r, mapsto] & H^0(V, \mathcal{F})^G\end{tikzcd}$$

For $V = pt$, recover group coh.

$X$ smooth variety $/\mathbb{R}$. Let $V = X(\mathbb{C})$.

Notation: $\mathbb{Z}(k) = (\sqrt{-1})^k \mathbb{Z}$ as a $G$-module ($\mathbb{Z}(2) = \mathbb{Z}$).
$M(k) = M \otimes_{\mathbb{Z}} \mathbb{Z}(k)$ for any $G$-module $M$.

Properties. • equivariant exponential sequence:

$$0 \to \mathbb{Z}(1) \xrightarrow{2\pi} \mathcal{O}_{X(\mathbb{C}), an} \xrightarrow{\exp} \mathcal{O}^*_{X(\mathbb{C}), an} \to 0$$

exact seq. of $G$-eq. sheaves.

induces

$$\begin{tikzcd}H^0_G(X(\mathbb{C}), \mathcal{O}^*_{an}) \arrow[r] & H^1_G(X(\mathbb{C}), \mathbb{Z}(1)) \\ \mathbb{R}^* \ni -1 \arrow[u, phantom, "\cup"] \arrow[r, mapsto] & \omega \arrow[u, phantom, "\Cup"]\end{tikzcd}$$

Def. $\omega \in H^1_G(X(\mathbb{C}), \mathbb{Z}(1))$ = the image of $-1$.

$\omega^k_X = \omega^k \in H^k_G(X(\mathbb{C}), \mathbb{Z}(k)) = \omega \cup \ldots \cup \omega$ ($k$ times).

$\omega^k_{X, \mathbb{Z}/2\mathbb{Z}} = \omega^k_{\mathbb{Z}/2\mathbb{Z}}$ := the image of $\omega^k$ by $H^k_G(X(\mathbb{C}), \mathbb{Z}(k)) \to H^k_G(X(\mathbb{C}), \mathbb{Z}/2\mathbb{Z}$

- equivariant GAGA: for proper $X$, $\mathrm{Pic}\, X = H^1_G(X(\mathbb{C}), \mathcal{O}^*_{an})$. [follows from GAGA]
- $M$ torsion $G$-module. $H^k(X(\mathbb{C}), M) = H^k_{ét}(X_{\mathbb{C}}, M)$
  $H^k_G(X(\mathbb{C}), M) = H^k_{ét}(X, M)$ (hence $H^k_G(X(\mathbb{C}), \mathbb{Z}(j)) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ $\overset{\shortparallel}{}$ $H^k_{ét}(X, \mathbb{Z}_\ell(j))$)

- there is a cycle class map

$$cl : CH^k(X) \longrightarrow H^{2k}_G(X(\mathbb{C}), \mathbb{Z}(k))$$

[compatible with products, push-forwards, pull-backs, etc...]
(for $k = 1$, $X$ proper: comes from eqv. exponential seq.)

- if $G$ acts freely on $X(\mathbb{C})$, i.e. if $X(\mathbb{R}) = \phi$, then $H^i_G(X(\mathbb{C}), M) = H^i(X(\mathbb{C})/G,$

→ if $X$ proper, smooth, irred., of dim. $d$, with $X(\mathbb{R}) = \phi$, get perfect Poincaré duality $H^i_G(X(\mathbb{C}), M) \times H^{2d-i}_G(X(\mathbb{C}), \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z}(d))) \to \mathbb{Q}/$

## 4) Curves of even geometric genus.

$B$ smooth proper geom. irred. curve $/\mathbb{R}$ with $B(\mathbb{R}) = \phi$.

$s(\mathbb{R}(B)) = 2$ i.e. $\exists\, \pi : B \longrightarrow \Gamma$, $\Gamma \subset \mathbb{P}^2_{\mathbb{R}} : x^2 + y^2 + z^2 = 0$.

How to tell whether $g(B)$ is even or odd ?!

**Lemma 4.1.** $g(B)$ is even iff $\deg(\pi)$ is odd.

**Proof**: exercise! [in $K'_0(\Gamma)$ : $[\pi_* \mathcal{O}_B] = \deg(\pi)[\mathcal{O}_\Gamma]$ + (supported on closed points)].

**Prop. 4.2.** $g(B)$ is even iff $\omega^2_{\mathbb{Z}/2\mathbb{Z}} \neq 0$ in $H^2_G(B(\mathbb{C}), \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.
└ Poincaré duality

**Proof**. • true if $B = \Gamma$ : $\Gamma(\mathbb{C})/G \simeq \mathbb{P}^2(\mathbb{R})$

$$\begin{tikzcd} H^1(\mathbb{P}^2(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \arrow[d, equal] & \times & H^1(\mathbb{P}^2(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \arrow[d, equal] \arrow[r] & \mathbb{Z}/2\mathbb{Z} \\ \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z}/2\mathbb{Z} & \end{tikzcd}$$

non-degenerate

so $\omega^2_{\mathbb{Z}/2\mathbb{Z}} \neq 0$.

and $\omega_{\mathbb{Z}/2\mathbb{Z}} \neq 0$ — obvious from defn.

$$\begin{tikzcd} \bullet\ \pi^* : H^2_G(\Gamma(\mathbb{C}), \mathbb{Z}/2\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^2_G(B(\mathbb{C}), \mathbb{Z}/2\mathbb{Z}) \arrow[d, equal] \\ \mathbb{Z}/2\mathbb{Z} \arrow[r, "\times \deg(\pi)"'] & \mathbb{Z}/2\mathbb{Z} \\ \omega^2_{\Gamma, \mathbb{Z}/2\mathbb{Z}} \arrow[r, mapsto] & \omega^2_{B, \mathbb{Z}/2\mathbb{Z}} \end{tikzcd}$$

let $X$ be a smooth, proper, irreducible variety of dim. $d$, such that $X(\mathbb{R}) = \phi$.

$H^{2d}_G(X(\mathbb{C}), \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ by Poincaré duality.

**Def.** let $\psi : H^{2d-2}_G(X(\mathbb{C}), \mathbb{Z}(d-1)) \longrightarrow \mathbb{Z}/2\mathbb{Z}$ be the map $\psi(x) = x \cup \omega^2_{X, \mathbb{Z}/2\mathbb{Z}}$

**Prop. 4.3.** let $B \subset X$ be an integral curve.

$$\psi(\mathrm{cl}(B)) = \begin{cases} 1 & \text{if } B \text{ is geom. ired. of even geometric genus} \\ 0 & \text{otherwise.} \end{cases}$$

**Proof**.

$$\begin{tikzcd} \tilde{B} \arrow[r] \arrow[rr, bend right, "\nu"'] & B \subset & X \end{tikzcd}$$

$$\nu_*(x \cup \underbrace{\nu^* \omega^2_{X, \mathbb{Z}/2\mathbb{Z}}}_{= \omega^2_{\tilde{B}, \mathbb{Z}/2\mathbb{Z}}}) = \nu_* x \cup \omega^2_{X, \mathbb{Z}/2\mathbb{Z}}$$

$\rightsquigarrow$ may assume $B = X$, $d = 1$.

$\rightsquigarrow$ aply prop. 4.2

## 5) Proof of Theorem 2.2.

$X$ smooth proper surface $/\mathbb{R}$. $X(\mathbb{R}) = \phi$.

**Prop. 5.1.** Then $\psi : H^2_G(X(\mathbb{C}), \mathbb{Z}(1)) \longrightarrow \mathbb{Z}/2\mathbb{Z}$ vanishes on $\mathrm{cl}(\mathrm{Pic}(X)_{\mathrm{tors}})$.

**Proof**. Claim. $\forall D \in \mathrm{Div}(X)$, $\psi(\mathrm{cl}(D)) = \chi(X, \mathcal{O}_X) - \chi(X, \mathcal{O}_X(-D)) \bmod 2 = \dfrac{D.(D+K)}{2} \bmod 2$

Claim $\Rightarrow$ Prop. : $D.(D+K) = 0$ if $[D]$ is torsion in $\mathrm{Pic}(X)$.

Proof of claim : • true if $D = B$ smooth curve by Prop. 4.3 $\big(\chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-D)) = \chi(\mathcal{O}_D) = 1 - g(B)\big)$

• $D \sim D' - D''$, $D', D''$ smooth ired. curves

$$\frac{1}{2}(D' - D'')(D' - D'' + K) = \frac{1}{2} D'(D' + K) - \frac{1}{2} D''(D'' + K) + \underbrace{(D'' - D').D''}_{\text{even (any 0-cycle on } X \text{ has even degree)}}$$

**Prop. 5.2.** $X$ smooth proper surface $/\mathbb{R}$, $X(\mathbb{R}) = \phi$.

Then $\omega^2_{\mathbb{Z}/2\mathbb{Z}} \in \mathrm{Im}\big(H^2_G(X(\mathbb{C}), \mathbb{Z}(1)) \longrightarrow H^2_G(X(\mathbb{C}), \mathbb{Z}/2\mathbb{Z})\big)$.

**Proof**.

$$\begin{tikzcd} & \mathrm{Pic}(X) \arrow[d] \arrow[r, leftrightarrow] & \mathrm{Pic}(X)_{\mathrm{tors}} \arrow[d, equal, "\text{— exercise!}"] \\ H^2_G(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(1)) & H^2_G(X(\mathbb{C}), \mathbb{Z}/2\mathbb{Z}) \arrow[l, "\beta"'] & H^1_G(X(\mathbb{C}), \mathbb{Q}/\mathbb{Z}(1)) \arrow[l, "\alpha"'] \\ \times & \times & \\ H^2_G(X(\mathbb{C}), \mathbb{Z}(1)) \arrow[r, "\gamma"] \arrow[d] & H^2_G(X(\mathbb{C}), \mathbb{Z}/2\mathbb{Z}) \arrow[d] & \\ \mathbb{Q}/\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \arrow[l, leftrightarrow] & \end{tikzcd}$$

$\ni \omega^2_{\mathbb{Z}/2\mathbb{Z}}$

$0 \to \mathbb{Z}/m\mathbb{Z}(1) \to \mathcal{O}^*_{an} \to \mathcal{O}^*_{an}$

$\omega^2_{\mathbb{Z}/2\mathbb{Z}} \in \mathrm{Im}(\gamma)$ iff $\omega^2_{\mathbb{Z}/2\mathbb{Z}}$ orthogonal to $\mathrm{Ker}(\beta) = \mathrm{Im}(\alpha)$ (Poincaré duality), which is true by Prop. 5.1

<u>Prop. 5.3.</u> If $H^2(X, \mathcal{O}_X) = 0$, then $\operatorname{Pic} X \twoheadrightarrow H^2_G(X(\mathbb{C}), \mathbb{Z}(1))$.

<u>Proof</u>: equv. exponential sequence.

Proof of Th. 2.2: $\omega^2_{\mathbb{Z}/2\mathbb{Z}} \in \operatorname{Im}(\operatorname{Pic} X \longrightarrow H^2_G(X(\mathbb{C}), \mathbb{Z}/2\mathbb{Z}))$

so for $U \subset X$ small enough, $\omega^2_{U, \mathbb{Z}/2\mathbb{Z}} = 0$ (in $H^2_G(U(\mathbb{C}), \mathbb{Z}/2\mathbb{Z})$

$$\begin{tikzcd} \omega^2_{U,\mathbb{Z}/2\mathbb{Z}} \arrow[d, mapsto] & H^2_G(U(\mathbb{C}), \mathbb{Z}/2\mathbb{Z}) \arrow[d, equal] \\ & H^2_{et}(U, \mathbb{Z}/2\mathbb{Z}) \arrow[d] \\ (-1,-1) \in \operatorname{Br} \mathbb{R}(X) \supset & H^2(\mathbb{R}(X), \mathbb{Z}/2\mathbb{Z}) \end{tikzcd}$$

so $(-1,-1) = 0$ in $\operatorname{Br} \mathbb{R}(X)$.

<u>so $s(\mathbb{R}(X)) \leq 2$.</u>

END OF SECOND LECTURE

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Exercise: X smooth proj. variety geom. irred. /ℝ, of dim ≥ 2.
If $X(\mathbb{R}) \neq \phi$, show X contains a curve of ~~odd~~ geom. genus and a curve of even $p_g$
$= \phi$, but need not
($\times 3$ or $C \times \mathbb{P}^1$)
odd genus

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## 6) <u>Looking for (rational) curves</u>

$X/\mathbb{C}$ smooth proper irreducible. dim $X = d$

<u>Def</u>. X is | <u>uniruled</u> if there exits a rational curve through a general point of X
| <u>rat. connected</u> (RC) .... two general points ... .

Example: unirational ⇒ RC.

<u>Theorem 6.1</u> (Kollár, Miyaoka, Mori (1992)) X RC ⇒ ∃ rat'l curve through any finite set of points of X.

<u>Conjecture 6.2</u> (Voisin) X RC $\Rightarrow H^{2d-2}(X(\mathbb{C}), \mathbb{Z})$ is generated by classes of (rational) curves.

$H^{2d-2}(X(\mathbb{C}), \mathbb{Z}) = \langle cl(\text{curves})\rangle \overset{\uparrow}{=} \langle cl(\text{rat. curves})\rangle$
Tian-Zong 2014.

<u>Analogous real questions</u>.

$X/\mathbb{R}$ smooth proper, irred, of dim. d.

<u>Q1</u> If $X_\mathbb{C}$ is RC, does X contain any <u>rational curve</u>?
↳:= geom. irred. of geom. genus 0

$B \subset X$ curve $\rightsquigarrow [B(\mathbb{R})] \in H^{d-1}(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$.

<u>Q2</u> If $X_\mathbb{C}$ is RC, is $H^{d-1}(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) = \langle cl(\text{curves})\rangle = \langle cl(\text{rat. curves})\rangle$ ?

<u>Theorem 6.3</u> (Kollár, 1999-2004). $X_\mathbb{C}$ RC, $d \geq 3$.
[ $m \geq 1$, $x_1, \dots, x_m \in X(\mathbb{R})$ in the same connected component.
Then ∃ smooth rational curve on X through $x_1, \dots, x_m$.
[∃? map $\Gamma \longrightarrow X$]

Says nothing on | Q1 when $X(\mathbb{R}) = \phi$ !
| Q2 (the proof constructs curves with contractible real locus)

<u>Remark</u>: Q1, Q2 open even for $X \subset \mathbb{P}^4_\mathbb{R}$ quartic threefold.
(Positive answer to Q1 would follow from Lang's conj. that $\mathbb{R}(\Gamma)$ is $C_1$
[conic ?!]

## 7) <u>Integral Hodge conjecture</u>

Example: $X \subset \mathbb{P}^{d+1}_\mathbb{C}$ smooth hypersurface of degree δ.

$\delta \leq N \Leftrightarrow X$ Fano $\Leftrightarrow X$ RC $\Leftrightarrow X$ uniruled.

$H^{2d-2}(X(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}$ (weak Lefschetz) $cl(B) \longmapsto \deg(B)$, $B \subset X$ curve

What is the subgroup of $\mathbb{Z}$ generated by algebraic classes (of curves)? difficult!

Say $N = 4$. For $\delta \leq 5$, $X$ contains a line $\rightsquigarrow$ get $\mathbb{Z}$.

<u>Conjecture 7.1</u> (Griffiths-Harris, 1985) For $\delta \geq 6$ and $X$ very general, get $\delta\mathbb{Z}$.

[like Noether-Lefschetz but Hodge theory will not help]

Kollár (1990): $\delta = 64$, $X$ very general $\Longrightarrow$ get a subgroup of $2\mathbb{Z}$.
"failure of the integral Hodge conjecture".

<u>Def.</u> A smooth proper irreducible $X/\mathbb{C}$ satisfies $IHC_1(X/\mathbb{C})$ if

$$CH_1(X) \twoheadrightarrow Hdg^{2d-2}(X(\mathbb{C}), \mathbb{Z}) = \{\alpha \in H^{2d-2}(X(\mathbb{C}), \mathbb{Z});\ \text{image of } \alpha \text{ in } H^{2d-2}(X(\mathbb{C}), \mathbb{C}) \text{ of type } (d-1, d-1)\}$$

(note $Hdg^{2d-2} = H^{2d-2}$ if $H^2(X, \mathcal{O}_X) = 0$)

($IHC_1$ holds for surfaces)

<u>Theorem 7.2</u> (Voisin, 2006) $IHC_1$ holds for
| uniruled 3-folds
| Calabi-Yau 3-folds.

(confirms Conj. 6.2 in dim. 3)

<u>Real formulation</u>.

$X$ smooth proper geom. irred. $/\mathbb{R}$ of dim. $d \geq 2$.

$$H_G^{2d-2}(X(\mathbb{C}), \mathbb{Z}(d-1)) \longrightarrow H_G^{2d-2}(X(\mathbb{R}), \mathbb{Z}(d-1)) = \bigoplus_{\substack{0 \leq p \leq d-1 \\ p \equiv d-1 \bmod 2}} H^p(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$$

$$\alpha \longmapsto (\alpha_p) \in$$

<u>Def.</u> $IHC_1(X/\mathbb{R})$ holds if

$$CH_1(X) \twoheadrightarrow Hdg_G^{2d-2}(X(\mathbb{C}), \mathbb{Z}(d-1))_0 := \{\alpha \in H_G^{2d-2}(X(\mathbb{C}), \mathbb{Z}(d-1));\ \text{image of } \alpha \text{ in } H^{2d-2}(X(\mathbb{C}), \mathbb{C}) \text{ has type } (d-1, d-1) \text{ and } \alpha_p = 0\ \forall p < d-1\}$$

"coh. with supports in degrees < codim support vanishes".

<u>Prop. 7.3</u>. $IHC_1(X/\mathbb{R})$ holds for surfaces. Proof: exp. exact seq.!

<u>Conjecture 7.4</u>. $IHC_1(X/\mathbb{R})$ holds if $X_\mathbb{C}$ is
| a uniruled 3-fold.
| a Calabi-Yau 3-fold
| RC (any dim.)

<u>Theorem 7.5</u> (B, W, 2016)

Conj. 7.4 holds for

- conic bundles over a surface (i.e. $X \to S$, generic fiber is a conic)
- Fano threefolds with $X(\mathbb{R}) = \phi$.

[also dP/curve]

## 8) <u>Consequences of IHC</u>.

Suppose $\pi_1(X(\mathbb{C}))^{ab} = 0$.

<u>Prop. 8.1</u>. $X/\mathbb{R}$ smooth proper geom. irred. of dim. $d$.

There is an exact sequence

$$\begin{tikzcd}
H^{2d-2}(X(\mathbb{C}), \mathbb{Z}) \arrow[r, "N_{\mathbb{C}/\mathbb{R}}"'] & H_G^{2d-2}(X(\mathbb{C}), \mathbb{Z}(d-1))_0 \arrow[r, "\psi"] & M \arrow[r] & 0 \\
CH_1(X_\mathbb{C}) \arrow[u] \arrow[r, "N_{\mathbb{C}/\mathbb{R}}"'] & CH_1(X) \arrow[u] \arrow[ur, bend right, "\varphi"'] & &
\end{tikzcd}$$

where

$$M = \begin{cases} H^{d-1}(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) \text{ and } \psi(\alpha) = \alpha_{d-1} & \text{if } X(\mathbb{R}) \neq \phi \\ \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \psi(\alpha) = \alpha \smile \omega^2_{\mathbb{Z}/2\mathbb{Z}} & \text{if } X(\mathbb{R}) = \phi. \end{cases}$$

(so $\varphi([B]) = \begin{cases} [B(\mathbb{R})] & \text{if } X(\mathbb{R}) \neq \phi \\ \text{detects genus mod } 2 \text{ (Prop. 4.3)} & \text{if } X(\mathbb{R}) = \phi. \end{cases}$

<u>Proof</u>: exercise when $X(\mathbb{R}) = \phi$. Hint: $0 \longrightarrow \mathbb{Z}(d) \longrightarrow \mathbb{Z}[G] \xrightarrow[N_{\mathbb{C}/\mathbb{R}}]{} \mathbb{Z}(d-1) \longrightarrow 0$.

<u>Cor. 8.2</u>. $X$ surface with $\pi_1(X(\mathbb{C}))^{ab} = 0$ and $H^2(X, \mathcal{O}_X) = 0$. Then $H^1 = \langle cl(\text{curves})\rangle$ and $\exists$ curve of even genus

Cor. 8.3. $X_{\mathbb{C}}$ RC 3-fold : $IHC_1(X/\mathbb{R}) \Longleftrightarrow H^{d-1}(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) = \langle cl(\text{curves})\rangle$ and $X$ contains a geom. ired. curve of even geometric genus.

Cor. 8.4 $X \subset \mathbb{P}^4_{\mathbb{R}}$ smooth hypersurface of degree $\delta$.

If $\delta = 3$, then $H^{d-1}(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z}) = \langle cl(\text{curves})\rangle$
$= \langle cl(\text{lines})\rangle$.

If $\delta = 4$ and $X(\mathbb{R}) = \phi$, then $X$ contains a curve of even genus.

Remarks:
- $IHC_1(X_{\mathbb{C}}/\mathbb{C})$ is trivial in the situation of Cor. 8.4
- for $X \subset \mathbb{P}^4_{\mathbb{R}}$ smooth quartic hyp. with $X(\mathbb{R}) = \phi$, one can compute that $H^4_G(X(\mathbb{C}), \mathbb{Z}(2)) = \underset{\substack{\uparrow \\ \text{class of } L+\bar{L}}}{\mathbb{Z}} \oplus \mathbb{Z}/2\mathbb{Z}$.
  $L \subset X_{\mathbb{C}}$ line

END OF THIRD LECTURE

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Exercise: find a surface $X/\mathbb{R}$ containing no geom. irreducible curve of even geom. genus (necessarily $X(\mathbb{R}) = \phi$)

Outline of proof of the existence of a curve of even genus on $X \subset \mathbb{P}^4_{\mathbb{R}}$ smooth quartic $X(\mathbb{R}) = \phi$ (Th. 7.5 + Cor. 8.3). Goal: $CH_1(X) \twoheadrightarrow H^4_G(X(\mathbb{C}), \mathbb{Z}(2))$.

[Voisin's strategy used for Th. 7.2]

① $S_0 \subset X$ smooth hyperplane section.

$H^2_G(S_0(\mathbb{C}), \mathbb{Z}(1)) \twoheadrightarrow H^4_G(X(\mathbb{C}), \mathbb{Z}(2))$ real weak Lefschetz.

② 
$$\begin{tikzcd} S \arrow[r] & (\mathbb{P}^4_{\mathbb{R}})^* \\ S_0 \arrow[u, phantom, sloped, "\subset"] \arrow[r] & 0 \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$
family of all hyp. sections

$U \subset (\mathbb{P}^4_{\mathbb{R}})^*(\mathbb{R})$ small nbd of $0$.

The $H^2_G(S_t(\mathbb{C}), \mathbb{Z}(1))$, $t \in U$ form a local system

Chose trivialisation $\rightsquigarrow$ trivialisation of vector bundle
$$\begin{tikzcd} H^2_G(\mathbb{R}(1)) = \bigcup_{t\in U} - \arrow[d] \\ U \end{tikzcd}$$

$$\begin{tikzcd} \lambda \in \mathrm{Hdg}^2_G(\mathbb{R}(1)) \arrow[r, phantom, "\subset"] \arrow[rr, bend left, "\tau"] \arrow[dd, mapsto] & H^2_G(\mathbb{R}(1)) \arrow[r] \arrow[dd] & H^2_G(S_0(\mathbb{C}), \mathbb{R}(1)) \arrow[d] & \\ & & H^2(S_0, \mathcal{O}_{S_0}) & H^2_G(S_0(\mathbb{C}), \mathbb{Z}(1)) \arrow[ul] \\ 0 \arrow[r, phantom, "\in"] & U & & \end{tikzcd}$$

If $\tau$ submersive for some $S_0$ (some $\lambda$) then $\mathrm{Im}(\tau) \supset$ open cone $\Longrightarrow H^2_G(S_0(\mathbb{C}), \mathbb{Z}(1))$ generated by classes which become Hodge on $S_t(\mathbb{C})$ for some $t \in U$ remain Hodge via action of monodromy

③ Otherwise: $d_\lambda \tau$ stabilises $\mathrm{Hdg}^2_G(S_0(\mathbb{C}), \mathbb{R}(1))$.

$(\dim H^2(S_0, \mathcal{O}_{S_0}) = 1.) \Rightarrow$ classes from $\mathrm{Hdg}^2_G(S_0(\mathbb{C}), \mathbb{R}(1))$ remain Hodge under all (small real) deformations.

$\Rightarrow$ contradict irred of monodromy on orthogonal to $[\mathcal{O}(1)]$.

(though [illegible] only care in $\mathrm{Hdg}^2(S_0(\mathbb{C})$

So if $\exists \alpha \in \mathrm{Hdg}^2_G(S_0(\mathbb{C}), \mathbb{R}(1))$ not multiple of $\mathcal{O}(1)$,
get $H^2(S_0(\mathbb{C}), \mathbb{R}(1)) = \mathrm{Hdg}^2(S_0(\mathbb{C}), \mathbb{R}(1))$.

Comment: link IHC $\longleftrightarrow$ curves of even genus works both ways (here, used Hodge theory as an indirect tool to prove $\exists$ curve of even genus; sometimes other way around: one wants to prove IHC (e.g. to control level) and one does this by exhibiting curve of even genus) (ex: double sextic 3-fold).

## 9) Level, coniveau, and the IHC.

Recall ~~open~~ question: $\exists$? real 3-fold $X$ with $s(\mathbb{R}(X)) = 8$ ?

Goal: explain that this is closely connected to IHC ($\rightsquigarrow$ motivation for $s$); explain candidate; higher-dim generalisation of theorem 2.2.

$F$ field of char. $\neq 2$. $H^m(F, \mathbb{Z}/2\mathbb{Z}) = H^m(\mathrm{Gal}(\overline{F}/F), \mathbb{Z}/2\mathbb{Z})$.

$(-1) \in F^*/F^{*2} = H^1(F, \mathbb{Z}/2\mathbb{Z})$. $(-1, \dots, -1) = (-1) \cup \dots \cup (-1)$.

**Theorem 9.1.** (Pfister, Elman, Lam, Voevodsky) $n \geqslant 1$. (more generally, isotropy of Pfister form $\leftrightarrow$ coh class $\neq 0$)

$s(F) < 2^m \iff (-1, \dots, -1) = 0 \in H^m(F, \mathbb{Z}/2\mathbb{Z})$.

[For $m=1$ clear; for $m=2$ already seen.]

$X/\mathbb{R}$ irred. variety ~~smooth proper~~ ~~$X(\mathbb{R}) = \phi$~~, $m \geqslant 1$, $M$ $G$-module

**Def.** $N^j H^m_G(X(\mathbb{C}), M) := \{\alpha \in H^m_G(X(\mathbb{C}), M);\ \exists Z \subseteq X$ closed subset of codim. $\geqslant j$ such that $\alpha|_{X \setminus Z} = 0 \in H^m_G((X \setminus Z)(\mathbb{C}), M)\}$.

"coniveau $\geqslant j$".

**Cor. 9.2.** $s(\mathbb{R}(X)) < 2^m \iff \omega^m_{\mathbb{Z}/2\mathbb{Z}}$ has coniveau $\geqslant 1$.

**Def.** $\mathcal{H}^q(M) :=$ the Zariski sheaf on $X$ associated to $U \longmapsto H^q_G(U(\mathbb{C}), M)$.

$\rightsquigarrow$ Leray spectral sequence for $X(\mathbb{C})/G \to X_{Zar}$: $(*)$ $E_2^{p,q} = H^p(X, \mathcal{H}^q(M)) \Rightarrow H^{p+q}_G(X(\mathbb{C}), M)$

**Theorem 9.3.** (Rost, Voevodsky) $\forall q$, $\mathcal{H}^q(\mathbb{Z}(q-1))_{tors} = 0$. (true over any field)

**Theorem 9.4.** (Bloch-Ogus, 1974) The filtration induced by $(*)$ is the coniveau filtration. $\rightsquigarrow$ to control coniveau, see where class goes in $(*)$: extremely useful. Application:

**Prop. 9.5.** (B.) $\omega^m_{\mathbb{Z}/2\mathbb{Z}}$ has coniveau $\geqslant 1 \Rightarrow \omega^{m+1}$ has coniveau $\geqslant 2$ $(\in H^{m+1}_G(X(\mathbb{C}), \mathbb{Z}(m+1))$

Also conversely if $CH_0(X_{\mathbb{C}})$ is supported in dim. $< m$ (e.g. $m = \dim X$ and $X_{\mathbb{C}}$ uniruled) (conjecturally iff $H^i(X, \mathcal{O}_X) = 0\ \forall i \geqslant m$)

Proof of $\Rightarrow$). $0 \to \mathbb{Z}(m+1) \xrightarrow[2]{} \mathbb{Z}(m+1) \to \mathbb{Z}/2\mathbb{Z} \to 0$ ~~$\rightsquigarrow$~~ $\rightsquigarrow H^m_G(X(\mathbb{C}), \mathbb{Z}/2\mathbb{Z}) \to H^{m+1}_G(X(\mathbb{C}), \mathbb{Z}(m+1))$

$\omega^m_{\mathbb{Z}/2\mathbb{Z}} \longmapsto \omega^{m+1}$

induces $\mathcal{H}^{m-1}(\mathbb{Z}/2\mathbb{Z}) \xrightarrow{0} \mathcal{H}^m(\mathbb{Z}(m+1))$ by Th. 9.3. | so $\omega^{m+1}$ ~~has~~ coniveau $\geqslant 1$.

induces $H^1(X, \mathcal{H}^{m-1}(\mathbb{Z}/2\mathbb{Z})) \longrightarrow H^1(X, \mathcal{H}^m(\mathbb{Z}(m+1)))$

$[\omega^m_{\mathbb{Z}/2\mathbb{Z}}] \longmapsto [\omega^{m+1}] = 0$ so $\omega^{m+1}$ has coniveau $\geqslant 2$.

($\Leftarrow$) generalizes Prop. 5.3: Prop. 5.2 $\iff$ for $d = 2$, $\omega^3 = 0$

Prop. 5.3 ~~said~~: for $d = 2$, $\omega^3 = 0 \Longrightarrow$ coniveau $\geqslant 1$ if $H^2(X, \mathcal{O}_X) = 0$)

Consequence:

$d = 3$. $s(\mathbb{R}(X)) < 8 \Longrightarrow \omega^4$ has coniveau $\geqslant 2$ ($\Longleftarrow$ if $X_{\mathbb{C}}$ is uniruled)

$\Updownarrow$

$\omega^4$ is algebraic i.e. $\in cl(CH_1(X))$

$IHC_1(X/\mathbb{R}) \Longrightarrow$ ($\omega^4$ is torsion $\Rightarrow$ Hodge).

$X(\mathbb{R}) = \phi$

$\omega^4 \in H^4_G(X(\mathbb{C}), \mathbb{Z}(4))$

$$\begin{tikzcd} H^4_G(X(\mathbb{C}), \mathbb{Z}(4)) \arrow[d, phantom, "\shortparallel"] & \\ H^4_G(X(\mathbb{C}), \mathbb{Z}(2)) & CH_1(X) \arrow[l, "cl"'] \end{tikzcd}$$

[combine with Conj. 7.4 !]

**Theorem 9.6** (BW, 2017)

$X$ real 3-fold, $X(\mathbb{R}) = \phi$, $X_{\mathbb{C}}$ uniruled.

Then $s(\mathbb{R}(X)) \leqslant 4$. (version of Th. 2.2 for 3-folds).

Suggestion: $X \subset \mathbb{P}^4_{\mathbb{R}}$ very general sextic hypersurface, $X(\mathbb{R}) = \phi$.

Then $s(\mathbb{R}(X)) = 8$ ?

Otherwise $\omega^4$ would be algebraic. One computes $H^4_G(X(\mathbb{C}), \mathbb{Z}(2)) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

"real Griffiths-Harris": $cl(CH_1(X)) = (3 \oplus 1).\mathbb{Z}$ ?!

$\omega^4 \longmapsto 0 \oplus 1$

[plane section] $\longmapsto 3 \oplus 1$

Proof of Th. 9.6. $X$ 3-fold, $X_{\mathbb{C}}$ uniruled, $X(\mathbb{R}) = \phi$.

Goal: $\omega^4$ has coniveau $\geq 2$. Proof combines Voisin's Th. 7.2 + rabbit

Theorem (Wu, 1955) Any compact smooth manifold $V$ of dim. $\leq 7$ satisfies

$$w_1^4 + w_2^2 + w_1 w_3 + w_4 = 0 \in H^4(V, \mathbb{Z}/2\mathbb{Z}).$$

[= 4th Wu class].

$\Downarrow$

$w_i$: i-th Stiefel-Whitney class.

$\rightsquigarrow$ express $w_i$ in terms of $c_j$ and $\omega$ $(V = X(\mathbb{C})/G)$. [started by Kahn in the 80's]

$$\Longrightarrow \omega^4_{\mathbb{Z}/2\mathbb{Z}} + \omega^2_{\mathbb{Z}/2\mathbb{Z}}.c_1 + c_1^2 + c_2 = 0 \quad \text{in } H^4_G(X(\mathbb{C}), \mathbb{Z}/2\mathbb{Z})$$

pour $d = 3$:
$w_1 = \omega$
$w_2 = \omega^2 + c_1$
$w_3 = \omega^3$
$w_4 = \omega^2 c_1 + c_2$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}(1) \arrow[r, "2"] & \mathbb{Z}(1) \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z}(1) \arrow[u, equal] \arrow[r, "{(1,-1)}"] & \mathbb{Z}[G] \arrow[u, "-"'] \arrow[r, "+", "N_{\mathbb{C}/\mathbb{R}}"'] & \mathbb{Z} \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

induces

$$\begin{tikzcd}
\overset{\omega^2.c_1}{\overset{\cap}{H^4_G(X(\mathbb{C}), \mathbb{Z}(1))}} \arrow[r] & H^4_G(X(\mathbb{C}), \mathbb{Z}/2\mathbb{Z}) \arrow[r] & H^5_G(X(\mathbb{C}), \mathbb{Z}(1)) \\
H^4(X(\mathbb{C}), \mathbb{Z}) \arrow[u] \arrow[r] & \underset{\omega^4 + c_1^2 + c_2}{\underset{\Psi}{H^4_G(X(\mathbb{C}), \mathbb{Z})}} \arrow[u] \arrow[r] & H^5_G(X(\mathbb{C}), \mathbb{Z}(1)) \arrow[u, equal]
\end{tikzcd}$$

have same image

$$\text{hence} \quad \omega^4 + \underbrace{c_1^2 + c_2}_{\text{coniveau} \geq 2} = N_{\mathbb{C}/\mathbb{R}}(\alpha) \quad \text{for some } \alpha \in \underbrace{H^4(X(\mathbb{C}), \mathbb{Z})}_{\text{coniveau} \geq 2}.$$

by Voisin (Th 7.2)

$\longleftarrow$

$\boxed{\text{If } H^2(X, \mathcal{O}_X) = 0}$

(otherwise conic bundle ...)

Analogue of $\underset{d=2}{\omega^3 = 0}$ (= Prop 5.2) is $\underset{d=3}{\omega^5 + \omega c_1^2 + \omega c_2 = 0}$

Dim. 4 again true that $\omega^5 + \omega c_1^2 + \omega c_2 = 0$ though not as easy!
then no need for Voisin: clearly $\omega^5$ has coniveau $\geq 2$.