Lecture 1. The Kuga–Satake variety

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Setting the scene: preliminaries on K3 surfaces

A complex (algebraic) K3 surface X is a smooth, projective, irreducible surface with $\Omega^2_X \cong \mathcal{O}_X$ and $\mathrm{H}^1(X, \mathcal{O}_X) = 0$. Using Serre duality and Riemann–Roch one finds that the classical (Betti) cohomology group $\mathrm{H}^2(X, \mathbb{Z})$ is a free abelian group of rank 22. We have the cup-product

$$\cup : \mathrm{H}^2(X, \mathbb{Z}) \times \mathrm{H}^2(X, \mathbb{Z}) \longrightarrow \mathrm{H}^4(X, \mathbb{Z}) \cong \mathbb{Z},$$

where the last isomorphism is due to the fact that $\dim(X) = 2$. This is a symmetric bilinear pairing. The Poincaré duality implies that this pairing is a perfect duality, that is, it induces an isomorphism

$$\mathrm{H}^2(X, \mathbb{Z}) \tilde{\longrightarrow} \mathrm{Hom}(\mathrm{H}^2(X, \mathbb{Z}), \mathbb{Z}).$$

In other words, the matrix of this bilinear form with respect to a $\mathbb{Z}$-basis of $\mathrm{H}^2(X, \mathbb{Z})$ has determinant in $\mathbb{Z}^\times = \{\pm 1\}$. Topological arguments (Wu's formula, Thom–Hirzebruch index theorem) give that the associated integral quadratic form is even, i.e. $(x^2) \in 2\mathbb{Z}$ for any $x \in \mathrm{H}^2(X, \mathbb{Z})$, and of signature $(3, 19)$. By the classification of even integral quadratic forms, this implies that $\mathrm{H}^2(X, \mathbb{Z})$ can written as the orthogonal direct sum $L = \mathrm{E}_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$. Here E_8 is the (positive definite) root lattice of the root system E_8; the lattice $\mathrm{E}_8(-1)$ is obtained by multiplication of the form on E_8 by -1, and U is the hyperbolic lattice of rank 2 (with quadratic form $2x_1x_2$).

Complex tori and their Hodge structures

Let M be a finitely generated free abelian group. Following Deligne, an integral HS on M is a representation of the 2-dimensional real torus $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ in $\mathrm{GL}(M_\mathbb{R})$. Then we have a Hodge decomposition $M_\mathbb{C} = \oplus_{p,q} M^{p,q}$ such that $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ acts on $M^{p,q}$ by $z^p \bar{z}^q$. The space $M^{q,p}$ is the complex conjugate of $M^{p,q}$. If $p + q = n$ for all terms in the decomposition, the HS is said to have weight n.

A complex torus is $\mathbb{C}^g/\Lambda$, where $\Lambda \cong \mathbb{Z}^{2g}$ is a full lattice, i.e. $\Lambda \otimes_\mathbb{Z} \mathbb{R} = \mathbb{C}^g$. To give a complex torus is the same as to give an integral Hodge structure of type $\{(1,0), (0,1)\}$ on Λ. An abelian variety is a complex torus with a polarisation, which is an integral skew-symmetric form on Λ satisfying some conditions. (This can be also rephrased by saying that the integral HS is polarisable.) For a curve C of genus g the spaces $\mathrm{H}^{1,0} \cong \mathrm{H}^0(C, \Omega^1_C)$ and $\mathrm{H}^{0,1} \cong \mathrm{H}^1(C, \mathcal{O}_C)$ have dimension g, so the Hodge decomposition

$$\mathrm{H}^1(C, \mathbb{Z})_{\mathbb{C}} = \mathrm{H}^1(C, \mathbb{C}) = \mathrm{H}^{1,0} \oplus \mathrm{H}^{0,1}$$

gives rise to a complex torus. Explicitly, integrating g linearly independent holomorphic 1-forms over $2g$ elements of a $\mathbb{Z}$-basis of $\mathrm{H}_1(C, \mathbb{Z})$ produces a full lattice $\Lambda \subset \mathbb{C}^g$ (defined up to a non-zero multiple). Then one shows that the complex torus $\mathbb{C}^g/\Lambda$ has a polarisation, so is an abelian variety.

Hodge structures of K3 type

In a very rough analogy to the Jacobian of a curve, one would like to associate to a complex K3 surface an abelian variety. So let us look at the Hodge structure of a K3 surface with a polarisation. The Hodge decomposition is

$$\mathrm{H}^2(X, \mathbb{Z})_{\mathbb{C}} = \mathrm{H}^2(X, \mathbb{C}) = \mathrm{H}^{2,0} \oplus \mathrm{H}^{1,1} \oplus \mathrm{H}^{0,2},$$

where $\mathrm{H}^{2,0} \cong \mathrm{H}^0(X, \Omega^2_X)$ and $\mathrm{H}^{0,2} \cong \mathrm{H}^2(X, \mathcal{O}_X)$ are both 1-dimensional vector spaces over $\mathbb{C}$. Choose a non-zero $\omega \in \mathrm{H}^{2,0}$. Since $\mathrm{H}^{4,0} = 0$ we have $(\omega^2) = 0$. The complex conjugate $\overline{\omega}$ is a non-zero element of $\mathrm{H}^{0,2}$. Since the pairing

$$\mathrm{H}^{2,0} \times \mathrm{H}^{0,2} \longrightarrow \mathrm{H}^{2,2} = \mathrm{H}^4(X, \mathbb{C}) \cong \mathbb{C}$$

is non-degenerate and the cup-product is symmetric, $(\omega.\overline{\omega})$ is a positive real number. Since $\mathrm{H}^{3,1} = 0$ we have $\mathrm{H}^{2,0} \perp \mathrm{H}^{1,1}$.

It is convenient to twist the HS on $\mathrm{H}^2(X, \mathbb{Z})$ by 1:

$$\mathrm{H}^2(X, \mathbb{Z}(1))_{\mathbb{C}} = \mathrm{H}^{1,-1} \oplus \mathrm{H}^{0,0} \oplus \mathrm{H}^{-1,1}.$$

This has the advantage that the image of $\mathbb{S}$ lies in $\mathrm{SO}(\mathrm{H}^2(X, \mathbb{Z}))_{\mathbb{R}}$. (This also means rescaling the image of the integral cohomology inside the complex cohomology by $2\pi i$, but we shall ignore this.)

The Picard group of a complex K3 surface is a free abelian gropup. Its rank ρ is called the Picard number. The cycle class map gives an embedding

$$\mathrm{Pic}(X) \hookrightarrow \mathrm{H}^2(X, \mathbb{Z}(1)).$$

By the Lefschetz theorem $\mathrm{Pic}(X) = \mathrm{H}^2(X, \mathbb{Z}(1)) \cap \mathrm{H}^{(0,0)}$. Hence $1 \leq \rho \leq 20$. The orthogonal complement to $\mathrm{Pic}(X)$ in $\mathrm{H}^2(X, \mathbb{Z}(1))$ is called the *transcendental lattice* and is denoted by $T(X)$.

Definition 0.1 *Let M be a finitely generated free abelian group with a non-degenerate integral symmetric bilinear form. An integral HS on M is called a* Hodge structure of K3 type*, if the Hodge decomposition is*

$$M_{\mathbb{C}} = M^{1,-1} \oplus M^{0,0} \oplus M^{-1,1},$$

where $\dim(M^{1,-1}) = 1$, $M^{1,-1} \perp M^{0,0}$, *and for a non-zero* $\omega \in M^{1,-1}$ *we have* $(\omega^2) = 0$, $(\omega.\overline{\omega}) > 0$.

Take a primitive element $\lambda \in L$, $(\lambda^2) = 2d > 0$. It can be proved that the set of such elements is an orbit of $\mathrm{Aut}(L)$, hence the isomorphism class of the orthogonal complement $\lambda^\perp \subset L$ depends only on d, e.g. $\lambda^\perp$ is isomorphic to

$$L_d = \mathrm{E}_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus (-2d).$$

L_d has signature $(2, 19)$. We thus associate to a K3 surface X with a primitive polarisation of degree $2d$ an integral HS of K3 type on L_d.

Associating to an integral HS on L_d of K3 type the 1-dimensional space $\mathrm{H}^{1,-1}$ defines a point in the *period domain*

$$\Omega_d = \{x \in \mathbb{P}(L_{d,\mathbb{C}}) \mid (x^2) = 0,\ (x, \bar{x}) > 0\} = \mathrm{SO}(2, 19)(\mathbb{R})/\mathrm{SO}(2)(\mathbb{R}) \times \mathrm{SO}(19)(\mathbb{R}).$$

The second formula identifies Ω_d with the Grassmannian of positive definite oriented 2-dimensional real subspaces of $L_d \otimes \mathbb{R} \simeq \mathbb{R}^{21}$, by sending x to the plane spanned by $\mathrm{Re}(x), \mathrm{Im}(x)$ in this order. Ω_d has two isomorphic connected components that are interchanged by the complex conjugation (or reversing the orientation).

Although the difference between HS of curves and K3 surfaces prevents us from constructing an analogue of the Jacobian for K3 surfaces without more work, we nevertheless have the following very important result. The classical Torelli theorem says that the isometry class of the integral HS on $\mathrm{H}^1(C, \mathbb{Z})$ uniquely determines the curve C. Piatetskii-Shapiro and Shafarevich proved that the isometry class of the integral HS on $\mathrm{H}^2(X, \mathbb{Z})$ uniquely determines the K3 surface X.

Another obstacle is that the cup-product pairing on $\mathrm{H}^2(X, \mathbb{Z})$ is symmetric, whereas for an abelian variety one would need a skew-symmetric pairing, such as the one given by the cup-product on $\mathrm{H}^1(C, \mathbb{Z})$. Is there a way to go from the special orthogonal group $\mathrm{SO}(21)$ to a simplectic group $\mathrm{Sp}(m)$ for some m? In the representation theory of Lie algebras one shows that the Lie algebra $\mathfrak{o}(2n + 1)$ has a remarkable irreducible representation of dimension 2^n, called the *spinor representation*. It has a unique invariant bilinear form which is skew-symmetric when n is 1 or 2 mod 4, so there is a way get a simplectic Lie algebra from $\mathfrak{o}(2n + 1)$.

Clifford algebra and spinor group

For Lie groups this is a bit more subtle: one needs to replace $\mathrm{SO}(21)$ (which is not simply connected) by its unramified double cover $\mathrm{Spin}(21) \to \mathrm{SO}(21)$. The spinor group is constructed using the Clifford algebra associated to our quadratic form. Let M be a finitely generated free abelian group with a non-degenerate quadratic form q. Define the Clifford algebra $\mathrm{C}(M)$ as the quotient of the full tensor algebra $\oplus_{n \geq 0} M^{\otimes n}$ by the two-sided ideal I generated by the elements of the form $x \otimes x - q(x)$, for $x \in M$. It is clear that there is an isomorphism of abelian groups $\mathrm{C}(M) \simeq \oplus_{n=0}^{\mathrm{rk}(M)} \wedge^n M$, so the rank of $\mathrm{C}(M)$ is $2^{\mathrm{rk}(M)}$. The multiplication by -1 on M acts on $\oplus_{n\geq 0}M^{\otimes n}$. Let us denote by plus the invariant elements and by minus the anti-invariant elements. Since $x\otimes x-q(x)$ is invariant, we have $I=I^+\oplus I^-$. Thus we can define

$$\mathrm{C}^+(M)=(\oplus_{n\geq 0}M^{\otimes 2n})/I^+,\quad \mathrm{C}^-(M)=(\oplus_{n\geq 0}M^{\otimes 2n+1})/I^-,$$

where the first equality is the quotient of a ring by an ideal, whereas the second one is the quotient of a (left or right) $\oplus_{n\geq 0}M^{\otimes 2n}$-module $\oplus_{n\geq 0}M^{\otimes 2n+1}$ by the submodule I^-. We have a natural homomorphism $M\to \mathrm{C}^-(M)$. Over an algebraically closed field the structure of the Clifford algebra is easy: if $\mathrm{rk}(M)$ is odd, then $\mathrm{C}^+(M_{\mathbb{C}})$ is isomorphic to a matrix algebra $\mathrm{End}_{\mathbb{C}}(W)$, where W is the spinor representation of $\mathrm{GSpin}(M_{\mathbb{C}})$. (Exercise: Prove that $\mathrm{C}(M_{\mathbb{C}})$ is isomorphic to a matrix algebra if $\mathrm{rk}(M)$ is even. For this assume that q is the orthogonal direct sum of the hyperbolic forms $(e_i^2)=(f_i^2)=0$, $(e_i,f_i)=1$, for $i=1,\ldots,n$, and show that W can be the full exterior algebra of the linear span of $e_1,\ldots,e_n$. Deduce that $\mathrm{C}^+(M_{\mathbb{C}})$ is isomorphic to a matrix algebra if $\mathrm{rk}(M)$ is odd.)

Define the *spinor group* as

$$\mathrm{GSpin}(M)=\{g\in \mathrm{C}^+(M)^\times | gMg^{-1}=M\}.$$

It acts by conjugation on M preserving the quadratic form. This gives an exact sequence of algebraic groups over $\mathbb{Q}$:

$$1\longrightarrow \mathbb{G}_{m,\mathbb{Q}}\longrightarrow \mathrm{GSpin}(M)_{\mathbb{Q}}\longrightarrow \mathrm{SO}(M)_{\mathbb{Q}}\longrightarrow 1.$$

Next, consider the *adjoint* action of $\mathrm{GSpin}(M)$ on $\mathrm{C}^+(M)$, i.e. the action by conjugations. This representation of $\mathrm{GSpin}(M)_{\mathbb{Q}}$ is isomorphic to the direct sum of $\wedge^{2n}M$ for $n\geq 0$.

Kuga–Satake construction I

Let us apply this to the second cohomology of a K3 surface X. Fix a primitive ample class $\lambda\in \mathrm{H}^2(X,\mathbb{Z}(1))$ and define P as the orthogonal complement to λ in $\mathrm{H}^2(X,\mathbb{Z}(1))$, in particular $\mathrm{rk}(P)=21$. We have

$$P_{\mathbb{C}}=P^{1,-1}\oplus P^{0,0}\oplus P^{-1,1}.$$

Kuga and Satake showed how to define a canonical complex structure on the real vector space $\mathrm{C}^+(P_{\mathbb{R}})$. We can normalise $\omega\in P^{1,-1}$ so that $(\omega.\overline{\omega})=2$. Write $\omega=\omega_1+i\omega_2$, where $\omega_1,\omega_2\in \mathrm{H}^2(X,\mathbb{R})$. Then $(\omega_1^2)=(\omega_2^2)=1$ and $(\omega_1,\omega_2)=0$. By the definition of the Clifford algebra, the following holds in $\mathrm{C}(P_{\mathbb{R}})$:

$$\omega_1^2=\omega_2^2=1,\quad \omega_1\omega_2=-\omega_2\omega_1.$$

Let $I=\omega_1\omega_2\in \mathrm{C}^+(P_{\mathbb{R}})$. (Check that I does not depend on ω.) Then $I^2=-1$, so the left multiplication by I defines a complex structure on the real vector space $\mathrm{C}^+(P_{\mathbb{R}})$, thus making $\mathrm{C}^+(P_{\mathbb{R}})/\mathrm{C}^+(P)$ a complex torus.

In Deligne's version one equips $\mathrm{C}^+(P)$ with an integral HS of type $\{(1,0),(0,1)\}$ as follows. Since $\mathbb{S}$ preserves the quadratic form on $P_{\mathbb{R}}$, we have a homomorphism $h:\mathbb{S}\to \mathrm{SO}(P)_{\mathbb{R}}$ whose kernel is $\{\pm 1\}$. For any $a,b\in\mathbb{R}$, not both equal to 0, we have $a+bI\in \mathrm{GSpin}(P)(\mathbb{R})$. Deligne points out that this is a canonical lifting of $h:\mathbb{S}\to \mathrm{SO}(P)_{\mathbb{R}}$ to $\tilde{h}:\mathbb{S}\hookrightarrow \mathrm{GSpin}(P)_{\mathbb{R}}$. (Exercise. Write $z=a+bi$. Check that $a+bI\in \mathrm{C}^+(P_{\mathbb{R}})$ and $x\mapsto (a+bI)x(a+bI)^{-1}$ acts on ω via multiplication by $z\bar{z}^{-1}$, on $\overline{\omega}$ via multiplication by $\bar{z}z^{-1}$, and fixes $P^{0,0}\cap P_{\mathbb{R}}$.) This means that the adjoint action of $\mathrm{GSpin}(P_{\mathbb{Q}})$ on P induces our original HS on P, so it is a HS of K3 type.

Lemma 0.2 *The left action of* $\mathrm{GSpin}(P)_{\mathbb{Q}}$ *on* $\mathrm{C}^+(P_{\mathbb{Q}})$ *induces an integral HS of type* $\{(1,0),(0,1)\}$ *on* $\mathrm{C}^+(P_{\mathbb{Q}})$.

Proof. The adjoint representation of $\mathrm{GSpin}(P)_{\mathbb{Q}}$ on $\mathrm{C}^+(P_{\mathbb{Q}})$ is isomorphic to the direct sum of $\wedge^{2n}P_{\mathbb{Q}}$ for $n\geq 0$. This implies that the induced HS on $\wedge^{2n}P_{\mathbb{Q}}$ is of type $\{(1,-1),(0,0),(-1,1)\}$. Hence the adjoint HS on $\mathrm{C}^+(P_{\mathbb{Q}})$ is also of type $\{(1,-1),(0,0),(-1,1)\}$. Over $\mathbb{C}$ the adjoint representation of $\mathrm{GSpin}(P)_{\mathbb{C}}$ on the matrix algebra $\mathrm{C}^+(P_{\mathbb{C}})$ is identified with $\mathrm{End}_{\mathbb{C}}(W)=W\otimes_{\mathbb{C}}W^*$, where W is the spinor representation. Thus the action of $\mathrm{GSpin}(P)_{\mathbb{C}}$ on $\mathrm{C}^+(P_{\mathbb{C}})$ by left multiplication is isomorphic to $W^{\dim(W)}$ as a representation of $\mathrm{GSpin}(P)_{\mathbb{C}}$. Therefore, the action of $\mathbb{S}\subset \mathrm{GSpin}(P)_{\mathbb{R}}$ by left multiplication induces an integral HS on $\mathrm{C}^+(P)$; its type must be $\{(1,0),(0,1)\}$ or $\{(-1,0),(0,-1)\}$ otherwise the HS on $W\otimes_{\mathbb{C}}W^*$ cannot be of type $\{(1,-1),(0,0),(-1,1)\}$. But $\mathbb{R}^\times\subset\mathbb{C}^\times$ acts on $\mathrm{C}^+(P)$ tautologically, so the weight of W is 1 and the type is $\{(1,0),(0,1)\}$. $\square$

It can be shown that this HS is polarisable, so we actually obtain an abelian variety and not just a complex torus. It is called the *Kuga–Satake variety* attached to (X,λ). Let us denote it by KS_X.