HW 9: Elliptische Kurven I

- Hand in by June 21st

Exercise 1. Prove or disprove:

1. The image of a morphism of affine varieties $f : X \to Y$ is a quasi-affine variety.
2. Let $f : X \to Y$ be a surjective morphism of connected affine curves[1]. Then the fibres of f are finite.

Exercise 2. Compute the singular locus of

1. the affine variety $X = Z(xyz)$ in $\mathbb{A}^3$,
2. the affine variety $Z(y^2 - x^3 - 5)$, and
3. the projective variety $Y = Z_+(x_0^2 + x_1^2 + x_2 x_3)$ in $\mathbb{P}^3$.

Exercise 3. Let $n \geq 0$ be an integer. Show that $\mathbb{P}^n$ is isomorphic to an affine variety if and only if $n = 0$.[2]

Exercise 4. Let X be a smooth projective irreducible curve. Let $f : X \to \mathbb{P}^1$ be a morphism of varieties.

1. Show that f is either constant or surjective.
2. Let U be the complement of $f^{-1}\{(1:0)\}$. Assume that U is non-empty. Show that $f|_U$, seen as a map from U to $\mathbb{A}^1 = \mathbb{P}^1 - \{(1:0)\}$, defines an element $\widetilde{f}$ of the field $K(X)$ of rational functions of X.
3. Show that $f \mapsto \widetilde{f}$ defines a bijection between $K(X)$ and the set of morphisms $X \to \mathbb{P}^1$ whose image is not $\{(1:0)\}$.
4. Let $X = \mathbb{P}^1$ and let $f : X \to \mathbb{P}^1$ be an isomorphism of projective varieties. Show that there exist $a, b, c, d \in k$ such that
$$\widetilde{f} = \frac{ax + b}{cx + d},$$
where we have identified $K(\mathbb{P}^1)$ with the field of fractions of $k[x] = \mathcal{O}_{\mathbb{P}^1}(\mathbb{A}^1)$. Deduce that $\mathrm{PGL}_2(k)$ is the group of automorphisms of $\mathbb{P}^1$.

Exercise 5. Show that, if X is an affine irreducible curve, then there is a surjective morphism $X \to \mathbb{P}^1$. [Hint: You may use that a non-constant morphism $X \to \mathbb{A}^1$ has open image.]

[1] An affine variety is a *curve* if all its irreducible components are one-dimensional.

[2] This exercise can be reformulated as follows. Let $n \geq 1$. Then there is no integer $m \geq 0$ such that $\mathbb{P}^n$ is isomorphic to a closed subset of $\mathbb{A}^m$ in the category of quasi-projective varieties (Vorlesung 11).