HW 6: Elliptische Kurven II

- Hand in by February 17th.

Exercise 1. Let E be an elliptic curve over $\mathbb{Q}$. Let L be a Galois extension of $\mathbb{Q}$ of *infinite* degree. Suppose that

$$\sup\{\mathrm{rank}(E(K)) \mid K \text{ a number field contained in } L\}$$

is finite.

1. Show that $E(L) \otimes \mathbb{Q}$ is a finite-dimensional $\mathbb{Q}$-vector space.
2. Prove that, if the torsion subgroup $E(L)_{\mathrm{tor}}$ of $E(L)$ is finite, then $E(L)$ is finitely generated.

Exercise 2. Prove or disprove:

1. If E is an elliptic curve over $\mathbb{Q}$, then the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the two-dimensional $\mathbb{F}_2$-vector space $E[2](\overline{\mathbb{Q}})$ is irreducible.
2. There is an elliptic curve E over $\mathbb{Q}$ with good reduction at the prime 2 such that $E(\mathbb{Q})_{tor}$ is of order eight.

Exercise 3. Let E be an elliptic curve given by a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with $a_1, \ldots, a_6 \in \mathbb{Z}$. Let $P \in E(\mathbb{Q})$ be a point of infinite order. Show that, if there is an integer $m \geq 1$ with $x([m]P) \in \mathbb{Z}$, then $x(P) \in \mathbb{Z}$.