HW 4: Elliptische Kurven II

- Hand in by February 3rd

Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. For $\alpha \in \overline{\mathbb{Q}}$ of degree $d \geq 1$ with minimal polynomial $f_\alpha \in \mathbb{Q}[x]$ given by $f_\alpha = a_0 + \ldots + a_{d-1}x^{d-1} + x^d$, we define its exponential height $H(\alpha)$ to be $\max(H(a_0), \ldots, H(a_{d-1}))$. We let $h(\alpha) = \log H(\alpha)$. Extend this height function h on $\overline{\mathbb{Q}}$ to a height function on $\mathbb{P}^1(\overline{\mathbb{Q}})$ by setting $h(\infty) = 0$. For E an elliptic curve over $\overline{\mathbb{Q}}$ given by $y^2 = x^3 + ax + b$ with a and b in $\overline{\mathbb{Q}}$ and for $P \in E(\overline{\mathbb{Q}}) \setminus \{0_E\}$, define $h(P) := h(x(P) : z(P))$. Set $h(0_E) = 0$.

Exercise 1. Prove or disprove:

1. For all $C \in \mathbb{R}$ and all $d \geq 1$, the set
$$\{\alpha \in \overline{\mathbb{Q}} \mid \dim_{\mathbb{Q}}(\mathbb{Q}(\alpha)) \leq d \text{ and } h(\alpha) \leq C\}$$
is finite.
2. There are only finitely many algebraic numbers $\alpha \in \overline{\mathbb{Q}}$ with $h(\alpha) = 0$.
3. If E is an elliptic curve over $\overline{\mathbb{Q}}$, then there are only finitely many points P in $E(\overline{\mathbb{Q}})$ such that $h(P) = 0$.
4. There is an elliptic curve E over $\mathbb{Q}$ such that the set of points P in $E(\mathbb{Q})$ with $\widehat{h}(P) = 0$ contains precisely one element.
5. There is an elliptic curve E over $\mathbb{Q}$ such that the set of points P in $E(\mathbb{Q})$ with $\widehat{h}(P) = 0$ contains precisely two elements.

Exercise 2. Explain how to prove Mordell-Weil for elliptic curves over a number field K (in as much detail as you desire), assuming weak Mordell-Weil for E over K.