HW 3: Elliptische Kurven II

- Hand in by December 6th.

Exercise 1. Prove or disprove:

1. If K and L are number fields with $O_K^* \cong O_L^*$ and $\mathrm{Cl}(K) \cong \mathrm{Cl}(L)$, then $K \cong L$.
2. There is an integer r such that, for all number fields K, the inequality

$$|d_K| \leq r$$

holds.

3. If K is a number field, then there is a finite extension L of K such that, for all ideals $\mathfrak{a}$ of O_K, the ideal $\mathfrak{a}O_L$ is principal.

Exercise 2. Prove or disprove:

1. A Dedekind domain with only finitely many prime ideals is a principal ideal domain.
2. If A is a Dedekind domain and $\mathfrak{a}$ is a nonzero ideal of A, then all ideals of the quotient ring $A/\mathfrak{a}$ are principal.
3. Every ideal of a Dedekind domain can be generated by two elements.
4. An integral domain in which all nonzero ideals admit a unique factorization into prime ideals is a Dedekind domain.

Exercise 3. Let $E = (E, 0_E)$ be an elliptic curve over $\mathbb{C}$. Let A be the coordinate ring of the smooth affine connected curve $E - \{0\}$.

1. Show that A is a Dedekind domain.
2. Let $\mathrm{Pic}(E)$ be the Picard group of E. Let $P \neq 0_E$ be a point of E with maximal ideal m_P in A. Show that the rule $P \mapsto m_P$ (for $P \neq 0_E$) and $0_E \mapsto 0$ induces a surjective homomorphism $\phi : \mathrm{Pic}(E) \to \mathrm{Cl}(A)$ with kernel isomorphic to $\mathbb{Z}$.
3. Show that $\mathrm{Cl}(A)$ is not finitely generated (hence infinite).