HW 1: Elliptische Kurven II

- Hand in by November 8th.

Exercise 1. Let E be the elliptic curve defined by $y^2 = x^3 - 7^2x$ over $\mathbb{C}$. Let $P = (0, 0)$ and $Q = (25, 120)$. Compute the following points:

1. $P + P$,
2. $P + Q$,
3. $P + P + Q$.

Exercise 2. Prove or disprove:

1. Let A be an abelian group. If A/nA is finite for all $n > 0$, then A is finitely generated.
2. If f and g are Weierstrass cubics with rational coefficients such that the elliptic curves over $\mathbb{C}$ defined by f and g are isomorphic, then the groups $E_f(\mathbb{Q})$ and $E_g(\mathbb{Q})$ are isomorphic.
3. If k is an algebraically closed field and E is an elliptic curve over k, then $E(k)$ is a finitely generated abelian group.
4. Let E be an elliptic curve over $\mathbb{Q}$ given by $y^2 = f(x)$ with $f(x) \in \mathbb{Q}[x]$. If K is a number field, then the rank of $E_f(K)$ equals the rank of $E_f(\mathbb{Q})$.

Exercise 3 (Optional)**.** Let E be an elliptic curve over $\mathbb{C}$. Let $E(\mathbb{C})$ be the group of points of E. Show that $E(\mathbb{C})$ contains a countably infinite subgroup H such that

1. H is a torsion group and $H/2H$ is trivial;
2. H is torsionfree and $H/2H$ is trivial;
3. H is torsionfree and $H/2H$ is infinite.