# HW 12: Elliptische Kurven I

- Hand in by July 26th.

Let $k$ be an algebraically closed field.

**Exercise 1.** Assume $2 \in k^\times$, and let $E$ be the elliptic curve given by $y^2 = x^3 - x$ over $k$.

1. Show that the map $[i] : E \to E$ given by $[i](x, y) = (-x, iy)$ defines an isogeny $[i] : E \to E$ and that $[i]$ satisfies $[i]^2 + [1] = 0$ in $\mathrm{End}(E)$.
2. Compute the $j$-invariant of $E$.

**Exercise 2.** Let $\zeta \in \mathbb{F}_4$ denote a 3rd root of unity. Let $E$ be the elliptic curve over $\overline{\mathbb{F}_4}$ defined by the equation $y^2 + y = x^3$. Let $f : E \to E$ be given by $f(x, y) = (\zeta x, y)$, and let $g : E \to E$ be given by $g(x, y) = (x + 1, y + x + \zeta)$

1. Show that $f$ and $g$ are automorphisms.
2. Show that $f$ and $g$ do not commute in the ring $\mathrm{End}(E)$, i.e., $f \circ g \neq g \circ f$.
3. Show that $\mathrm{rk}_{\mathbb{Z}}(\mathrm{End}(E)) \geq 4$.

**Exercise 3.** Prove or disprove:

1. If $k = \overline{\mathbb{F}_p}$, then there exists an elliptic curve $E$ over $k$ such that $\mathrm{End}(E) = \mathbb{Z}$.
2. If $E$ and $E'$ are elliptic curves over $k$, then there is an isogeny from $E$ to $E'$.
3. If $E$ is an elliptic curve over $k$, then the set of isomorphism classes of elliptic curves $E'$ over $k$ such that $E$ is isogenous to $E'$ is finite.