HW 11: Elliptische Kurven I

- Hand in by July 12th.

The base field is an algebraically closed field k.

Exercise 1. Prove or disprove:

1. Let X and Y be smooth projective irreducible curves. Suppose that there is a quasi-projective curve U such that U is isomorphic to some open of X and some open of Y. Then X is isomorphic to Y.
2. The set of isomorphism classes of smooth projective irreducible curves over $\mathbb{C}$ is finite.
3. The set of equivalence classes of rational functions $f \in \mathbb{C}(t)$ of degree d is finite. (Here f and g in $\mathbb{C}(t)$ are *equivalent* if there exists an automorphism σ of $\mathbb{P}^1$ such that $f\sigma = g\sigma$ as morphisms from $\mathbb{P}^1 \to \mathbb{P}^1$.)
4. Let F be a homogeneous polynomial in $k[x_0, \ldots, x_n]$ of degree $d \geq 1$. For $i \in \{0, \ldots, n\}$, let F_i be the partial derivative of F with respect to x_i. If $d \neq 0$ in k, then the ideal $(F_0, \ldots, F_n)$ equals the ideal $(F, F_0, \ldots, F_n)$ in $k[x_0, \ldots, x_n]$.
5. **A one-dimensional irreducible closed subset of $\mathbb{A}^3$ is the intersection of two hypersurfaces.**

Exercise 2. Let X be a smooth projective irreducible curve and let P be a point in X. Use Riemann-Roch to see that $\mathcal{O}_X(X - \{P\})$ is not a finite-dimensional k-vector space.

Exercise 3. Let k be an algebraically closed field with $7 \in k^\times$.

1. Compute the singular locus of the projective curve C given by the homogeneous equation $x^3y + y^3z + z^3x = 0$ in $\mathbb{P}^2(k)$.
2. Show that C has non-trivial automorphisms.
3. Compute the divisors of the functions x/y and x/z.

Let X be a smooth projective irreducible curve. Let $f : X \to \mathbb{P}^1$ be a surjective morphism. We define $\deg f$ of f to be the degree of the divisor

$$(f)_Q := \sum_{P \in X, f(P)=Q} v_P(f)[P],$$

where $Q \in \mathbb{P}^1$ is any point.

Exercise 4. Show that $\deg(f)$ is a well-defined positive integer, i.e., show that $\deg(f)$ is independent of the choice of $Q \in \mathbb{P}^1$.

Exercise 5. Let X be a smooth projective irreducible curve of genus g over k. Prove or disprove (by means of a counterexample):

1. There exists a non-constant rational function $f : X \to \mathbb{P}^1(k)$ such that $\deg f \leq g + 1$.
2. The degree of the divisor of a non-zero rational 1-form on X only depends on the genus g of X.
3. If D is a non-zero effective divisor on X, then $\dim_k \mathrm{H}^0(X, D) = \deg D + 1 - g$.

Exercise 6. Let k be an algebraically closed field. Let X be a smooth projective irreducible curve of genus 2 over k with canonical divisor K_X. For D a divisor on X, prove that

$$\dim_k \mathrm{H}^0(X, D) = \begin{cases} \deg D - 1 & \deg D \geq 3 \\ 2 & D \equiv K_X \\ 1 & \deg D = 2, D \not\equiv K_X \\ 0 & \deg D < 0 \end{cases}.$$

(Here $D \equiv K_X$ means that $D - K_X$ is the divisor of some non-zero rational function on X.)

Exercise 7. Assume $6 \in k^*$. Let a and b be elements of k such that $4a^3 + 27b^2 \neq 0$. Let E be the smooth projective irreducible cubic curve in $\mathbb{P}^2$ associated to the affine Weierstrass equation $y^2 = x^3 + ax + b$. Let $0_E = (0 : 1 : 0)$ be the point at infinity.

1. What are the coordinates of all the non-trivial 2-torsion points on E? (A point P is 2-torsion on E if $P + P = 0_E$.)
2. How many 3-torsion points does E have?
3. Assume $a = 0$ and $b = 1$. Compute $(2, 3) + (2, 3)$ in E.
4. Assume $a = 0$ and $b = 1$. Can you compute $6 \cdot (2, 3)$?