EXAM: Elliptische Kurven II

- Hand in by March 3rd.

Exercise 1. In this exercise, we let $k = \overline{\mathbb{F}_2}$.

1. How many k-isomorphism classes of elliptic curves which can be defined over $\mathbb{F}_2$ are there?
2. Write down (generalized) Weierstrass equations for all (isomorphism classes of) elliptic curves over k which can be defined over $\mathbb{F}_2$.

Exercise 2. Let p be an odd prime number.

1. Show that the discriminant of the elliptic curve E_p defined by $y^2 + xy = x^3 + p$ is minimal at p. [Does E_p have nodal reduction at p?]
2. Show that E_3, E_5 and E_7 have positive rank.

Exercise 3. For each integer m, let $N(m) = \#\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid y^2 = x^3 + m\}$.

1. Is $N(m)$ finite for all $m \in \mathbb{Z}$?
2. Show that $N(m)$ can be arbitrarily large as m runs over $\mathbb{Z}$. [Hint: Use that there is an $m_0 \in \mathbb{Z}$ such that $y^2 = x^3 + m_0$ has positive rank.)
3. Use a suitable height function to show that, there is a real number $c > 0$ (independent of m) such that, for infinitely many integers m, the inequality $N(m) > c(\log |m|)^{1/3}$ holds.

Exercise 4. Let E be the elliptic curve $y^2 + y = x^3 - x^2$ over $\mathbb{Q}$.

1. Compute $E(\mathbb{Q})_{tor}$.
2. Show that $E(\mathbb{Q})$ has rank zero.

Exercise 5. Show that there are elliptic curves E_1 and E_2 over $\mathbb{Q}$ such that there is an isogeny from E_1 to E_2 defined over $\mathbb{Q}$ and $\Delta_{min}(E_1) \neq \Delta_{min}(E_2)$.

Exercise 6. Compute all solutions of the equation $y^2 = x^3 - x$ with x and y in $\mathbb{Z}$.

Exercise 7. Let E be the elliptic curve defined by $y^2 + y = x^3 - x$. Assume as given that $E(\mathbb{Q})$ has rank one.

1. Prove that $E(\mathbb{Q}) \cong \mathbb{Z}$.

2. Use Exercise 4 from HW6 to show that $(0,0) \in E(\mathbb{Q})$ is a generator for $E(\mathbb{Q})$.

3. Compute the set of solutions of the equation $y^2 + y = x^3 - x$ with x and y in $\mathbb{Z}$. [Hint: Let $P = (0,0)$. Suppose $[m]P$ is integral. Write $m = 2^a n$ with n odd, and use Exercise 4 from HW6 to show that $[n]P$ is integral. Use an argument as in (2) to find all possible values for n, and then do some computations to find the possible a's.]

4. Find all positive integers which are simultaneously the product of two consecutive positive integers and the product of three consecutive positive integers.