Arakelov invariants of Belyi curves

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Pour mon amie, Ami

Contents

Introduction

Let $\mathbf{Q} \to \overline{\mathbf{Q}}$ be an algebraic closure of the field of rational numbers $\mathbf{Q}$. In this thesis we obtain explicit bounds for Arakelov invariants of curves over $\overline{\mathbf{Q}}$. We use our results to give algorithmic, geometric and Diophantine applications.

Let X be a smooth projective connected curve over $\overline{\mathbf{Q}}$ of genus g. Belyi proved that there exists a finite morphism $X \to \mathbf{P}^1_{\overline{\mathbf{Q}}}$ ramified over at most three points; see [5]. Let $\deg_B(X)$ denote the Belyi degree of X, i.e., the minimal degree of a finite morphism $X \to \mathbf{P}^1_{\overline{\mathbf{Q}}}$ unramified over $\mathbf{P}^1_{\overline{\mathbf{Q}}} - \{0, 1, \infty\}$. Since the topological fundamental group of the projective line $\mathbf{P}^1(\mathbf{C})$ minus three points is finitely generated, the set of $\overline{\mathbf{Q}}$-isomorphism classes of curves with bounded Belyi degree is finite.

We prove that, if $g \geq 1$, the Faltings height $h_{\mathrm{Fal}}(X)$, the Faltings delta invariant $\delta_{\mathrm{Fal}}(X)$, the discriminant $\Delta(X)$ and the self-intersection of the dualizing sheaf $e(X)$ are bounded by a polynomial in $\deg_B(X)$; the precise definitions of these Arakelov invariants of X are given in Section 1.5.

Theorem A. *For any smooth projective connected curve X over $\overline{\mathbf{Q}}$ of positive genus g,*

$$\begin{array}{ccccc} -\log(2\pi)g & \leq & h_{\mathrm{Fal}}(X) & \leq & 13 \cdot 10^6 g \deg_B(X)^5 \\ 0 & \leq & e(X) & \leq & 3 \cdot 10^7 (g-1) \deg_B(X)^5 \\ 0 & \leq & \Delta(X) & \leq & 5 \cdot 10^8 g^2 \deg_B(X)^5 \\ -10^8 g^2 \deg_B(X)^5 & \leq & \delta_{\mathrm{Fal}}(X) & \leq & 2 \cdot 10^8 g \deg_B(X)^5. \end{array}$$

We give several applications of Theorem A in this thesis. Before we explain these, let us mention that the Arakelov invariants $h_{\mathrm{Fal}}(X)$, $e(X)$, $\Delta(X)$ and $\delta_{\mathrm{Fal}}(X)$ in Theorem A all have a different flavour to them. For example, the Faltings height $h_{\mathrm{Fal}}(X)$ plays a key role in Faltings' proof of his finiteness theorem on abelian varieties; see [23]. On the other hand, the strict positivity of $e(X)$ (when $g \geq 2$) is related to the Bogomolov conjecture; see [64]. The discriminant $\Delta(X)$ "measures" the bad reduction of the curve $X/\overline{\mathbf{Q}}$, and appears in Szpiro's discriminant conjecture for semi-stable elliptic curves; see [62]. Finally, as was remarked by Faltings in his introduction to [24], Faltings' delta invariant $\delta_{\mathrm{Fal}}(X)$ can be viewed as the minus logarithm of a "distance" to the boundary of the moduli space of compact connected Riemann surfaces of genus g.

We were first led to investigate this problem by work of Edixhoven, de Jong and Schepers on covers of complex algebraic surfaces with fixed branch locus; see [22]. They conjectured an arithmetic analogue ([22, Conjecture 5.1]) of their main theorem (Theorem 1.1 in *loc. cit.*). We use our results to prove this conjecture; see Section 3.3 for a more precise statement.

Let us briefly indicate where the reader can find some applications of Theorem A in this thesis.

1. The Couveignes-Edixhoven-Bruin algorithm for computing coefficients of modular forms runs in polynomial time under the Riemann hypothesis for number fields; see Section 3.1.

2. Let U be a smooth quasi-projective curve over $\overline{\mathbf{Q}}$. We show that the "height" of a finite étale cover of degree d of U is bounded by a polynomial in d; see Section 3.3.

3. Theorem A gives explicit bounds for the "complexity" of the semi-stable reduction of a curve in terms of its Belyi degree. From this, we obtain explicit bounds on the "complexity" of the semi-stable reduction for modular curves, Fermat curves and Galois Belyi curves; see Corollary 3.2.2.

4. We prove a conjecture of Szpiro for genus g curves X over a number field K with fixed set S of bad reduction (*Szpiro's small points conjecture*) in a special case. More precisely, we prove Szpiro's small points conjecture for cyclic covers of prime degree; see Theorem 4.4.1.

In the course of proving Theorem A we establish several results which will certainly interest some readers.

– We show that, in order to bound Arakelov invariants of a curve X over $\overline{\mathbf{Q}}$, it essentially suffices to find an algebraic point x in $X(\overline{\mathbf{Q}})$ of bounded height; see Theorem 2.2.1.
– We prove a generalization of Dedekind's discriminant conjecture; we learned the argument from H.W. Lenstra jr. (Section 2.4.1).
– We use a theorem of Merkl-Bruin to prove explicit bounds for Arakelov-Green functions of Belyi covers; see Section 2.3.
– We use techniques due to Q. Liu and D. Lorenzini to construct suitable models for covers of curves; see Theorem 2.4.9.

To prove Theorem A we will use Arakelov theory for curves over a number field K. To apply Arakelov theory in this context, we will work with *arithmetic surfaces* associated to such curves, i.e., regular projective models over the ring of integers O_K of K. We refer the reader to Section

1.2 for precise definitions. For a smooth projective connected curve X over $\overline{\mathbf{Q}}$ of genus $g \geq 1$, we define the Faltings height $h_{\mathrm{Fal}}(X)$, the discriminant $\Delta(X)$, Faltings' delta invariant $\delta_{\mathrm{Fal}}(X)$ and the self-intersection of the dualizing sheaf $e(X)$ in Section 1.5. These are the four Arakelov invariants appearing in Theorem A.

We introduce two functions on $X(\overline{\mathbf{Q}})$ in Section 1.7: the canonical Arakelov height function and the Arakelov norm of the Wronskian differential. We show that, to prove Theorem A, it suffices to bound the canonical height of some non-Weierstrass point and the Arakelov norm of the Wronskian differential at this point; see Theorem 2.2.1 for a precise statement.

We estimate Arakelov-Green functions and Arakelov norms of Wronskian differentials on finite étale covers of the modular curve $Y(2)$ in Theorem 2.3.12 and Proposition 2.3.13, respectively. In our proof we use an explicit version of a result of Merkl on the Arakelov-Green function; see Theorem 2.3.2. This version of Merkl's theorem was obtained by Peter Bruin in his master's thesis; see [9]. The proof of this version of Merkl's theorem is reproduced in the appendix to [30] by Peter Bruin.

In Section 2.5.2 we prove the existence of a non-Weierstrass point on X of bounded height; see Theorem 2.5.4. The proof of Theorem 2.5.4 relies on our bounds for Arakelov-Green functions (Theorem 2.3.12), the existence of a "wild" model (Theorem 2.4.9) and a generalization of Dedekind's discriminant conjecture for discrete valuation rings of characteristic zero (Proposition 2.4.1) which we attribute to Lenstra.

A precise combination of the above results constitutes the proof of Theorem A given in Section 2.5.3.

The main result of this thesis (Theorem A) also appears in our paper [30]. In *loc. cit.* the reader can also find the applications of Theorem A given in Chapter 3. The proof of Szpiro's small points conjecture for cyclic covers of prime degree is joint work with Rafael von Känel; see [31].

CHAPTER 1

Arakelov invariants, canonical Arakelov height, Belyi degree

We are going to apply Arakelov theory to smooth projective geometrically connected curves X over number fields K. In [3] Arakelov defined an intersection theory on the *arithmetic surfaces* attached to such curves. In [24] Faltings extended Arakelov's work. In this chapter we aim at giving the necessary definitions for what we need later (and we need at least to fix our notation).

We start with some preparations concerning Riemann surfaces and arithmetic surfaces; see Section 1.1 and Section 1.2. We recall some basic properties of semi-stable arithmetic surfaces in Section 1.4. In Section 1.5 we define the main objects of study of this thesis: Arakelov invariants of curves over $\overline{\mathbf{Q}}$. For the sake of completeness, we also included a section on Arakelov invariants of abelian varieties (Section 1.6). The results of that section will not be used to prove the main result of this thesis. To prove the main result of this thesis, we will work with the canonical Arakelov height function on a curve over $\overline{\mathbf{Q}}$; see Section 1.7. A crucial ingredient is an upper bound for the Faltings height in terms of the height of a non-Weierstrass point and the Arakelov norm of the Wronskian differential; this is the main result of Section 1.8. Finally, we introduce the Belyi degree in Section 1.9 and prove some of its basic properties.

1.1. Arakelov invariants of Riemann surfaces

In this sections we follow closely [21, Section 4.4]. Let X be a compact connected Riemann surface of genus $g \geq 1$. The space of holomorphic differentials $\mathrm{H}^0(X, \Omega^1_X)$ carries a natural hermitian inner product:

$$(\omega, \eta) \quad \mapsto \quad \frac{i}{2} \int_X \omega \wedge \overline{\eta}.$$

For any orthonormal basis $(\omega_1, \ldots, \omega_g)$ with respect to this inner product, the Arakelov $(1,1)$-form is the smooth positive real-valued $(1,1)$-form μ on X given by

$$\mu = \frac{i}{2g} \sum_{k=1}^{g} \omega_k \wedge \overline{\omega_k}.$$

Note that μ is independent of the choice of orthonormal basis. Moreover, $\int_X \mu = 1$.

Denote by $\mathcal{C}^\infty$ the sheaf of complex valued C^∞-functions on X, and by $\mathcal{A}^1$ the sheaf of complex C^∞ 1-forms on X. There is a tautological differential operator $d\colon \mathcal{C}^\infty \to \mathcal{A}^1$. It decomposes as $d = \partial + \overline{\partial}$ where, for any local C^∞ function f and any holomorphic local coordinate z, with real and imaginary parts x and y, one has $\partial f = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}) \cdot dz$ and $\overline{\partial} f = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}) \cdot d\overline{z}$.

Proposition 1.1.1. *For each a in X, there exists a unique real-valued g_a in $\mathcal{C}^\infty(X - \{a\})$ such that the following properties hold:*

1. *we can write $g_a = \log|z - z(a)| + h$ in an open neighbourhood of a, where z is a local holomorphic coordinate and where h is a C^∞-function;*
2. *$\partial\overline{\partial} g_a = \pi i \mu$ on $X - \{a\}$;*
3. *$\int_X g_a \mu = 0$.*

Let gr_X be the Arakelov-Green function on $(X \times X)\backslash\Delta$, where $\Delta \subset X \times X$ denotes the diagonal. That is, for any a and b in X, we have $\mathrm{gr}_X(a,b) = g_a(b)$ with g_a as in Proposition 1.1.1; see [3], [15], [21] or [24] for a further discussion of the Arakelov-Green function gr_X. The Arakelov-Green functions determine certain metrics whose curvature forms are multiples of μ, called *admissible metrics*, on all line bundles $\mathcal{O}_X(D)$, where D is a divisor on X, as well as on the holomorphic cotangent bundle Ω^1_X. Explicitly: for $D = \sum_P D_P P$ a divisor on X (with D_P a real number), the metric $\|\cdot\|$ on $\mathcal{O}_X(D)$ satisfies $\log\|1\|(Q) = \mathrm{gr}_X(D,Q)$ for all Q away from the support of D, where

$$\mathrm{gr}_X(D,Q) := \sum_P D_P \, \mathrm{gr}_X(P,Q).$$

Furthermore, for a local coordinate z at a point a in X, the metric $\|\cdot\|_{\mathrm{Ar}}$ on the sheaf Ω^1_X satisfies

$$-\log\|dz\|_{\mathrm{Ar}}(a) = \lim_{b \to a}\left(\mathrm{gr}_X(a,b) - \log|z(a) - z(b)|\right).$$

We will work with these metrics on $\mathcal{O}_X(P)$ and Ω^1_X (as well as on tensor product combinations of them) and refer to them as *Arakelov metrics*. A metrised line bundle $\mathcal{L}$ is called *admissible* if, up to a constant scaling factor, it is isomorphic to one of the admissible bundles $\mathcal{O}_X(D)$. Note that it is non-trivial to show that the line bundle Ω^1_X endowed with the above metric is admissible; see [3] for details. For an admissible line bundle $\mathcal{L}$, we have $\mathrm{curv}(\mathcal{L}) = (\deg\mathcal{L}) \cdot \mu$ by Stokes' theorem.

For any admissible line bundle $\mathcal{L}$, we endow the determinant of cohomology

$$\lambda(\mathcal{L}) = \det \mathrm{H}^0(X, \mathcal{L}) \otimes \det \mathrm{H}^1(X, \mathcal{L})^\vee$$

of the underlying line bundle with the Faltings metric, i.e., the metric on $\lambda(L)$ determined by the following set of axioms (cf. [24]): (i) any isometric isomorphism $\mathcal{L}_1 \xrightarrow{\sim} \mathcal{L}_2$ of admissible line bundles induces an isometric isomorphism $\lambda(\mathcal{L}_1) \xrightarrow{\sim} \lambda(\mathcal{L}_2)$; (ii) if we scale the metric on $\mathcal{L}$ by a factor α, the metric on $\lambda(\mathcal{L})$ is scaled by a factor $\alpha^{\chi(\mathcal{L})}$, where

$$\chi(\mathcal{L}) = \deg \mathcal{L} - g + 1$$

is the Euler-Poincaré characteristic of $\mathcal{L}$; (iii) for any divisor D and any point P on X, the exact sequence

$$0 \to \mathcal{O}_X(D - P) \to \mathcal{O}_X(D) \to P_* P^* \mathcal{O}_X(D) \to 0$$

induces an isometry $\lambda(\mathcal{O}_X(D)) \xrightarrow{\sim} \lambda(\mathcal{O}_X(D-P)) \otimes P^*\mathcal{O}_X(D)$; (iv) for $\mathcal{L} = \Omega^1_X$, the metric on $\lambda(\mathcal{L}) \cong \det \mathrm{H}^0(X, \Omega^1_X)$ is defined by the hermitian inner product

$$(\omega, \eta) \mapsto (i/2) \int_X \omega \wedge \overline{\eta}$$

on $\mathrm{H}^0(X, \Omega^1_X)$. In particular, for an admissible line bundle $\mathcal{L}$ of degree $g-1$, the metric on the determinant of cohomology $\lambda(\mathcal{L})$ does not depend on the scaling.

Let $\mathbf{H}_g$ be the Siegel upper half space of complex symmetric g-by-g-matrices with positive definite imaginary part. Let τ in $\mathbf{H}_g$ be the period matrix attached to a symplectic basis of $\mathrm{H}_1(X, \mathbf{Z})$ and consider the analytic Jacobian

$$J_\tau(X) = \mathbf{C}^g / (\mathbf{Z}^g + \tau \mathbf{Z}^g)$$

attached to τ. On $\mathbf{C}^g$ one has a theta function

$$\vartheta(z;\tau) = \vartheta_{0,0}(z;\tau) = \sum_{n \in \mathbf{Z}^g} \exp(\pi i \, {}^t n \tau n + 2\pi i \, {}^t n z),$$

giving rise to a reduced effective divisor Θ_0 and a line bundle $\mathcal{O}(\Theta_0)$ on $J_\tau(X)$. The function ϑ is not well-defined on $J_\tau(X)$. Instead, we consider the function

$$\|\vartheta\|(z;\tau) \quad = \quad (\det \Im(\tau))^{1/4} \exp(-\pi \, {}^t y (\Im(\tau))^{-1} y) |\vartheta(z;\tau)|,$$

with $y = \Im(z)$. One can check that $\|\vartheta\|$ descends to a function on $J_\tau(X)$. Now consider on the other hand the set $\mathrm{Pic}_{g-1}(X)$ of divisor classes of degree $g-1$ on X. It comes with a canonical subset Θ given by the classes of effective divisors and a canonical bijection $\mathrm{Pic}_{g-1}(X) \xrightarrow{\sim} J_\tau(X)$ mapping Θ onto Θ_0. As a result, we can equip $\mathrm{Pic}_{g-1}(X)$ with the structure of a compact complex manifold, together with a divisor Θ and a line bundle $\mathcal{O}(\Theta)$. Note that we obtain $\|\vartheta\|$

as a function on $\mathrm{Pic}_{g-1}(X)$. It can be checked that this function is independent of the choice of τ. Furthermore, note that $\|\vartheta\|$ gives a canonical way to put a metric on the line bundle $\mathcal{O}(\Theta)$ on $\mathrm{Pic}_{g-1}(X)$.

For any line bundle $\mathcal{L}$ of degree $g-1$ there is a canonical isomorphism from $\lambda(\mathcal{L})$ to $\mathcal{O}(-\Theta)[\mathcal{L}]$, the fibre of $\mathcal{O}(-\Theta)$ at the point $[\mathcal{L}]$ in $\mathrm{Pic}_{g-1}(X)$ determined by $\mathcal{L}$. Faltings proves that when we give both sides the metrics discussed above, the norm of this isomorphism is a constant independent of $\mathcal{L}$; see [24, Section 3]. We will write this norm as $\exp(\delta_{\mathrm{Fal}}(X)/8)$ and refer to $\delta_{\mathrm{Fal}}(X)$ as Faltings' delta invariant of X. (Note that $\delta_{\mathrm{Fal}}(X)$ was denoted as $\delta(X)$ by Faltings in [24].)

Let $S(X)$ be the real number defined by

$$\log S(X) \quad = \quad -\int_X \log \|\vartheta\|(gP-Q)\cdot \mu(P), \tag{1.1.1}$$

where Q is any point on X; see [15]. It is related to Faltings' delta invariant $\delta_{\mathrm{Fal}}(X)$. In fact, let $(\omega_1,\ldots,\omega_g)$ be an orthonormal basis of $\mathrm{H}^0(X,\Omega_X^1)$. Let b be a point on X and let z be a local coordinate about b. Write $\omega_k = f_k dz$ for $k=1,\ldots,g$. We have a holomorphic function

$$W_z(\omega) = \det\left(\frac{1}{(l-1)!}\frac{d^{l-1}f_k}{dz^{l-1}}\right)_{1\leq k,l\leq g}$$

locally about b from which we build the $g(g+1)/2$-fold holomorphic differential

$$W_z(\omega)(dz)^{\otimes g(g+1)/2}.$$

It is readily checked that this holomorphic differential is independent of the choice of local coordinate and orthonormal basis. Thus, this holomorphic differential extends over X to give a non-zero global section, denoted by Wr, of the line bundle $\Omega_X^{\otimes g(g+1)/2}$. The divisor of the non-zero global section Wr, denoted by $\mathcal{W}$, is the divisor of Weierstrass points. This divisor is effective of degree g^3-g. We follow [15, Definition 5.3] and denote the constant norm of the canonical isomorphism of (abstract) line bundles

$$\Omega_X^{g(g+1)/2}\otimes_{\mathcal{O}_X}\left(\Lambda^g \mathrm{H}^0(X,\Omega_X^1)\otimes_{\mathbf{C}}\mathcal{O}_X\right)^{\vee} \longrightarrow \mathcal{O}_X(\mathcal{W})$$

by $R(X)$. Then,

$$\log S(X) \quad = \quad \frac{1}{8}\delta_{\mathrm{Fal}}(X) + \log R(X). \tag{1.1.2}$$

Moreover, for any non-Weierstrass point b in X,

$$\mathrm{gr}_X(\mathcal{W},b) - \log R(X) \quad = \quad \log\|\mathrm{Wr}\|_{\mathrm{Ar}}(b). \tag{1.1.3}$$

1.2. Arakelov invariants of arithmetic surfaces

Let K be a number field with ring of integers O_K, and let $S = \operatorname{Spec} O_K$. Let $p : \mathcal{X} \to S$ be an arithmetic surface, i.e., an integral regular flat projective S-scheme of relative dimension 1 with geometrically connected fibres; see [41, Chapter 8.3] for basic properties of arithmetic surfaces.

Suppose that the genus of the generic fibre $\mathcal{X}_K$ is positive. An Arakelov divisor D on $\mathcal{X}$ is a divisor D_{fin} on $\mathcal{X}$, plus a contribution $D_{\mathrm{inf}} = \sum_\sigma \alpha_\sigma F_\sigma$ running over the embeddings $\sigma : K \longrightarrow \mathbf{C}$ of K into the complex numbers. Here the α_σ are real numbers and the F_σ are formally the "fibers at infinity", corresponding to the Riemann surfaces $\mathcal{X}_\sigma$ associated to the algebraic curves $\mathcal{X} \times_{O_K,\sigma} \mathbf{C}$. We let $\widehat{\mathrm{Div}}(\mathcal{X})$ denote the group of Arakelov divisors on $\mathcal{X}$. To a non-zero rational function f on $\mathcal{X}$, we associate an Arakelov divisor $\widehat{\mathrm{div}}(f) := (f)_{\mathrm{fin}} + (f)_{\mathrm{inf}}$ with $(f)_{\mathrm{fin}}$ the usual divisor associated to f on $\mathcal{X}$, and $(f)_{\mathrm{inf}} = \sum_\sigma v_\sigma(f) F_\sigma$, where we define $v_\sigma(f) := -\int_{\mathcal{X}_\sigma} \log |f|_\sigma \cdot \mu_\sigma$. Here μ_σ is the Arakelov $(1,1)$-form on $\mathcal{X}_\sigma$ as in Section 1.1. We will say that two Arakelov divisors on $\mathcal{X}$ are linearly equivalent if their difference is of the form $\widehat{\mathrm{div}}(f)$ for some non-zero rational function f on $\mathcal{X}$. We let $\widehat{\mathrm{Cl}}(\mathcal{X})$ denote the group of Arakelov divisors modulo linear equivalence on $\mathcal{X}$.

In [3] Arakelov showed that there exists a unique symmetric bilinear map

$$(\cdot,\cdot) : \widehat{\mathrm{Cl}}(\mathcal{X}) \times \widehat{\mathrm{Cl}}(\mathcal{X}) \longrightarrow \mathbf{R}$$

with the following properties:

– if D and E are effective divisors on $\mathcal{X}$ without common component, then

$$(D,E) = (D,E)_{\mathrm{fin}} - \sum_{\sigma : K \to \mathbf{C}} \mathrm{gr}_{\mathcal{X}_\sigma}(D_\sigma, E_\sigma),$$

where σ runs over the complex embeddings of K. Here $(D,E)_{\mathrm{fin}}$ denotes the usual intersection number of D and E as in [41, Section 9.1], i.e.,

$$(D,E)_{\mathrm{fin}} = \sum_{s \in |S|} i_s(D,E) \log \# k(s),$$

where s runs over the set of closed points $|S|$ of S, $i_s(D,E)$ is the intersection multiplicity of D and E at s and $k(s)$ denotes the residue field of s. Note that if D or E is vertical ([41, Definition 8.3.5]), the sum $\sum_{\sigma : K \to \mathbf{C}} \mathrm{gr}_{\mathcal{X}_\sigma}(D_\sigma, E_\sigma)$ is zero;

– if D is a horizontal divisor ([41, Definition 8.3.5]) of generic degree n over S, then $(D, F_\sigma) = n$ for every $\sigma : K \longrightarrow \mathbf{C}$;

– if $\sigma_1, \sigma_2 : K \to \mathbf{C}$ are complex embeddings, then $(F_{\sigma_1}, F_{\sigma_2}) = 0$.

In particular, if D is a vertical divisor and $E = E_{\text{fin}} + E_{\text{inf}}$ is an Arakelov divisor on $\mathcal{X}$, we have $(D, E) = (D, E_{\text{fin}})_{\text{fin}}$.

An *admissible line bundle* on $\mathcal{X}$ is the datum of a line bundle $\mathcal{L}$ on $\mathcal{X}$, together with admissible metrics on the restrictions $\mathcal{L}_\sigma$ of $\mathcal{L}$ to the $\mathcal{X}_\sigma$. Let $\widehat{\mathrm{Pic}}(\mathcal{X})$ denote the group of isomorphism classes of admissible line bundles on $\mathcal{X}$. To any Arakelov divisor $D = D_{\text{fin}} + D_{\text{inf}}$ with $D_{\text{inf}} = \sum_\sigma \alpha_\sigma F_\sigma$, we can associate an admissible line bundle $\mathcal{O}_{\mathcal{X}}(D)$. In fact, for the underlying line bundle of $\mathcal{O}_{\mathcal{X}}(D)$ we take $\mathcal{O}_{\mathcal{X}}(D_{\text{fin}})$. Then, we make this into an admissible line bundle by equipping the pull-back of $\mathcal{O}_{\mathcal{X}}(D_{\text{fin}})$ to each $\mathcal{X}_\sigma$ with its Arakelov metric, multiplied by $\exp(-\alpha_\sigma)$. This induces an isomorphism $\widehat{\mathrm{Cl}}(\mathcal{X}) \xrightarrow{\sim} \widehat{\mathrm{Pic}}(\mathcal{X})$. In particular, the Arakelov intersection of two admissible line bundles on $\mathcal{X}$ is well-defined.

Recall that a metrised line bundle $(\mathcal{L}, \|\cdot\|)$ on $\operatorname{Spec} O_K$ corresponds to an invertible O_K-module, L, say, with hermitian metrics on the complex vector spaces $L_\sigma := \mathbf{C} \otimes_{\sigma, O_K} L$. The *Arakelov degree* of $(\mathcal{L}, \|\cdot\|)$ is the real number defined by:

$$\widehat{\deg}(\mathcal{L}) = \widehat{\deg}(\mathcal{L}, \|\cdot\|) = \log \#(L/O_K s) - \sum_{\sigma\colon K \to \mathbf{C}} \log \|s\|_\sigma,$$

where s is any non-zero element of L (independence of the choice of s follows from the product formula).

Note that the relative dualizing sheaf $\omega_{\mathcal{X}/O_K}$ of $p : \mathcal{X} \to S$ is an admissible line bundle on $\mathcal{X}$ if we endow the restrictions $\Omega^1_{\mathcal{X}_\sigma}$ of $\omega_{\mathcal{X}/O_K}$ to the $\mathcal{X}_\sigma$ with their Arakelov metric. Furthermore, for any section $P : S \to \mathcal{X}$, we have

$$\widehat{\deg}\, P^* \omega_{\mathcal{X}/O_K} = (\mathcal{O}_X(P), \omega_{\mathcal{X}/O_K}) =: (P, \omega_{\mathcal{X}/O_K}),$$

where we endow the line bundle $P^* \omega_{\mathcal{X}/O_K}$ on $\operatorname{Spec} O_K$ with the pull-back metric.

We state three basic properties of Arakelov's intersection pairing; see [3] and [24].

Adjunction formula: Let $b : \operatorname{Spec} O_K \to \mathcal{X}$ be a section. Then

$$(b, b) = -(\mathcal{O}_{\mathcal{X}}(b), \omega_{\mathcal{X}/O_K}),$$

where we identify $b : \operatorname{Spec} O_K \to \mathcal{X}$ with its image in $\mathcal{X}$.

Base change: Let L/K be a finite field extension with ring of integers O_L, and let

$$q : \operatorname{Spec} O_L \to \operatorname{Spec} O_K$$

be the associated morphism. Then, if $\mathcal{X}' \to \mathcal{X} \times_{O_K} \operatorname{Spec} O_L$ denotes the minimal resolution of singularities and $r : \mathcal{X}' \to \mathcal{X}$ is the associated morphism, for two admissible line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on $\mathcal{X}$,

$$(r^* \mathcal{L}_1, r^* \mathcal{L}_2) = [L : K](\mathcal{L}_1, \mathcal{L}_2).$$

Riemann-Roch: Let $\mathcal{L}$ be an admissible line bundle on $\mathcal{X}$. Let $\det R^{\cdot}p_*\mathcal{L}$ be the determinant of cohomology on $\operatorname{Spec} O_K$ endowed with the Faltings metric (defined in Section 1.1). Then there is a canonical isomorphism of metrized line bundles

$$\det R^{\cdot}p_*\omega_{\mathcal{X}/O_K} = \det p_*\omega_{\mathcal{X}/O_K}$$

on $\operatorname{Spec} O_K$ and

$$\widehat{\deg} \det R^{\cdot}p_*L = \frac{1}{2}(\mathcal{L}, \mathcal{L}\otimes\omega_{\mathcal{X}/O_K}^{-1}) + \widehat{\deg} \det p_*\omega_{\mathcal{X}/O_K}.$$

We are now ready to define certain invariants (read "real numbers") associated to the arithmetic surface $p : \mathcal{X} \to \operatorname{Spec} O_K$. We will refer to these invariants as *Arakelov invariants* of $\mathcal{X}$.

The *Faltings delta invariant* of $\mathcal{X}$ is defined as

$$\delta_{\mathrm{Fal}}(\mathcal{X}) = \sum_{\sigma:O_K\to\mathbf{C}} \delta_{\mathrm{Fal}}(\mathcal{X}_\sigma),$$

where σ runs over the complex embeddings of O_K into $\mathbf{C}$. Similarly, we define

$$\|\vartheta\|_{\max}(\mathcal{X}) = \prod_{\sigma:O_K\to\mathbf{C}} \max_{\mathrm{Pic}_{g-1}(\mathcal{X}_\sigma)} \|\vartheta\|.$$

Moreover, we define

$$R(\mathcal{X}) = \prod_{\sigma:O_K\to\mathbf{C}} R(\mathcal{X}_\sigma), \quad S(X) = \prod_{\sigma:O_K\to\mathbf{C}} S(\mathcal{X}_\sigma).$$

The *Faltings height* of $\mathcal{X}$ is defined by

$$h_{\mathrm{Fal}}(\mathcal{X}) = \widehat{\deg} \det p_*\omega_{\mathcal{X}/O_K} = \widehat{\deg} \det R^{\cdot}p_*\mathcal{O}_{\mathcal{X}},$$

where we endow the determinant of cohomology with the Faltings metric (Section 1.1) and applied Serre duality. Furthermore, we define the *self-intersection of the dualizing sheaf* of $\mathcal{X}$, denoted by $e(\mathcal{X})$, as

$$e(\mathcal{X}) := (\omega_{\mathcal{X}/O_K}, \omega_{\mathcal{X}/O_K}),$$

where we employed Arakelov's intersection pairing on the arithmetic surface $\mathcal{X}/O_K$.

1.3. Arakelov invariants of curves over number fields

Let K be a number field with ring of integers O_K. For a curve X over K, a *regular (projective) model* of X over O_K consists of the data of an arithmetic surface $p : \mathcal{X} \to \operatorname{Spec} O_K$ and an isomorphism $X \cong \mathcal{X}_\eta$ of the generic fibre $\mathcal{X}_\eta$ of $p : \mathcal{X} \to \operatorname{Spec} O_K$ over K. Recall that any smooth projective geometrically connected curve X over K has a regular model by Lipman's theorem ([41, Theorem 9.3.44]). For a curve X over K, a *(relatively) minimal regular model* of X over O_K is a regular model $p : \mathcal{X} \to \operatorname{Spec} O_K$ which does not contain any exceptional divisors; see [41, Definition 9.3.12]. Any smooth projective geometrically connected curve over K of positive genus admits a unique minimal regular model over O_K; see [41, Theorem 9.3.21].

Let X be a smooth projective geometrically connected curve over K of positive genus. We define certain invariants (read "real numbers") associated to X. We will refer to these invariants of X as *Arakelov invariants*.

Let $p : \mathcal{X} \to \operatorname{Spec} O_K$ be the minimal regular model of X over O_K. Then

$$\delta_{\mathrm{Fal}}(X/K) := \delta_{\mathrm{Fal}}(\mathcal{X}), \quad \|\vartheta\|_{\max}(X/K) := \|\vartheta\|_{\max}(\mathcal{X}),$$

$$S(X/K) := S(\mathcal{X}), \quad R(X/K) := R(\mathcal{X}).$$

Moreover,

$$h_{\mathrm{Fal}}(X/K) := h_{\mathrm{Fal}}(\mathcal{X}), \quad e(X/K) := e(\mathcal{X}).$$

The following proposition shows that the Arakelov invariant $h_{\mathrm{Fal}}(X/K)$ can be computed on any regular model of X over O_K.

Proposition 1.3.1. *Let $\mathcal{Y} \to \operatorname{Spec} O_K$ be a regular model for X over O_K. Then $h_{\mathrm{Fal}}(X/K) = h_{\mathrm{Fal}}(\mathcal{Y})$.*

Proof. Recall that $p : \mathcal{X} \to \operatorname{Spec} O_K$ denotes the minimal regular model of X over O_K. By the minimality of $\mathcal{X}$, there exists a unique birational morphism $\phi : \mathcal{Y} \to \mathcal{X}$; see [41, Corollary 9.3.24]. Let E be the exceptional locus of ϕ. Since the line bundles $\omega_{\mathcal{Y}/O_K}$ and $\phi^*\omega_{\mathcal{X}/O_K}$ agree on $\mathcal{Y}_i - E$, there is an effective vertical divisor V (supported on E) and an isomorphism of admissible line bundles

$$\omega_{\mathcal{Y}/O_K} = \phi^*\omega_{\mathcal{X}/O_K} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{Y}(V).$$

By the projection formula and the equality $\phi_*\mathcal{O}_\mathcal{Y}(V) = \mathcal{O}_\mathcal{X}$, we obtain that

$$(p\phi)_*\omega_{\mathcal{Y}/O_K} = p_*\phi_*(\phi^*\omega_{\mathcal{X}/O_K} \otimes_{\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{Y}(V)) = p_*\omega_{\mathcal{X}/O_K}.$$

In particular, $\det(p\phi)_*\omega_{\mathcal{Y}/O_K} = \det p_*\omega_{\mathcal{X}/O_K}$. Taking the Arakelov degree, the latter implies that

$$h_{\mathrm{Fal}}(X/K) = h_{\mathrm{Fal}}(\mathcal{X}) = h_{\mathrm{Fal}}(\mathcal{Y}).$$

□

1.4. Semi-stability

The Arakelov invariants of curves we introduce in this chapter are associated to models with "semi-stable" fibers. In this short section, we give the necessary definitions and basic properties needed in this thesis concerning semi-stable arithmetic surfaces.

Let K be a number field with ring of integers O_K.

Definition 1.4.1. Let $p : \mathcal{X} \to \operatorname{Spec} O_K$ be an arithmetic surface. We say that $\mathcal{X}$ is *semi-stable (or nodal) over* O_K if every geometric fibre of $\mathcal{X}$ over O_K is reduced and has only ordinary double singularities; see [41, Definition 10.3.1].

Remark 1.4.2. Suppose that $\mathcal{X}$ is semi-stable and minimal. The blowing-up $\mathcal{Y} \to \mathcal{X}$ along a smooth closed point on $\mathcal{X}$ is semi-stable over O_K, but no longer minimal.

Definition 1.4.3. Let $p : \mathcal{X} \to \operatorname{Spec} O_K$ be a semi-stable arithmetic surface. The *discriminant* of $\mathcal{X}$ (over O_K), denoted by $\Delta(\mathcal{X})$, is defined as

$$\Delta(\mathcal{X}) = \sum_{\mathfrak{p} \subset O_K} \delta_{\mathfrak{p}} \log \# k(\mathfrak{p}),$$

where $\mathfrak{p}$ runs through the maximal ideals of O_K and $\delta_{\mathfrak{p}}$ denotes the number of singularities in the geometric fibre of $p : \mathcal{X} \to \operatorname{Spec} O_K$ over $\mathfrak{p}$. Since $p : \mathcal{X} \to \operatorname{Spec} O_K$ is smooth over all but finitely many points in $\operatorname{Spec} O_K$ and the fibres of $\mathcal{X} \to \operatorname{Spec} O_K$ are geometrically reduced, the discriminant of $\mathcal{X}$ is a well-defined real number.

We will work with the following version of the semi-stable reduction theorem.

Theorem 1.4.4. (Deligne-Mumford) [18] *Let X be a smooth projective geometrically connected curve over K of positive genus. Then, there exists a finite field extension L/K such that the minimal regular model of the curve X_L over O_L is semi-stable over O_L.*

Theorem 1.4.5. *Let $p : \mathcal{X} \to \operatorname{Spec} O_K$ be a semi-stable arithmetic surface. Let L/K be a finite field extension, and let O_L be the ring of integers of L. Let $\mathcal{X}' \to \mathcal{X} \times_{O_K} O_L$ be the minimal resolution of singularities, and let $r : \mathcal{X}' \to \mathcal{X}$ be the induced morphism.*

1. *The arithmetic surface $p' : \mathcal{X}' \to \operatorname{Spec} O_L$ is semi-stable.*
2. *The equality of discriminants $\Delta(\mathcal{X})[L : K] = \Delta(\mathcal{X}')$ holds.*
3. *The canonical morphism $\omega_{\mathcal{X}'/O_L} \to r^*\omega_{\mathcal{X}/O_K}$ is an isomorphism of line bundles on $\mathcal{X}'$.*
4. *The equality $e(\mathcal{X})[L : K] = e(\mathcal{X}')$ holds.*

5. *Let* $q : \operatorname{Spec} O_L \to \operatorname{Spec} O_K$ *be the morphism of schemes associated to the inclusion* $O_K \subset O_L$. *Then, the canonical map*

$$\det p'_* \omega_{\mathcal{X}'/O_L} \to q^* \det p_* \omega_{\mathcal{X}/O_K}$$

is an isomorphism of line bundles on $\operatorname{Spec} O_L$.

6. *The equality* $h_{\mathrm{Fal}}(\mathcal{X})[L : K] = h_{\mathrm{Fal}}(\mathcal{X}')$ *holds.*

Proof. We start with the first two assertions. To prove these, we note that the scheme $\mathcal{X} \times_{O_K} O_L$ is normal and each geometric fibre of the flat projective morphism $\mathcal{X} \times_{O_K} O_L \to \operatorname{Spec} O_L$ is connected, reduced with only ordinary double singularities. Thus, the minimal resolution of singularities $\mathcal{X}' \to \mathcal{X} \times_{O_K} O_L$ is obtained by resolving the double points of $\mathcal{X} \times_{O_K} O_L$. By [41, Corollary 10.3.25], a double point in the fiber of $\mathcal{X} \otimes_{O_K} O_L \to \operatorname{Spec} O_L$ over the maximal ideal $\mathfrak{q} \subset O_L$ is resolved by $e_{\mathfrak{q}} - 1$ irreducible components of multiplicity 1 isomorphic to $\mathbf{P}^1_{k(\mathfrak{q})}$ with self-intersection -2, where $k(\mathfrak{q})$ denotes the residue field of $\mathfrak{q}$ and $e_{\mathfrak{q}}$ is the ramification index of $\mathfrak{q}$ over O_K. This proves the first two assertions. The third assertion is proved in [37, Proposition V.5.5]. The fourth assertion follows from the third assertion and basic properties of Arakelov's intersection pairing. Finally, note that (5) follows from (3) and (6) follows from (5). □

Definition 1.4.6. Let X be a smooth projective geometrically connected curve over K with semi-stable reduction over O_K, and let $\mathcal{X} \to \operatorname{Spec} O_K$ be its minimal regular (semi-stable) model over O_K. We define the *discriminant* of X over K by $\Delta(X/K) := \Delta(\mathcal{X})$.

Remark 1.4.7. Let us mention that, more generally, one can define the "relative discriminant" of a curve X over K to be the Artin conductor of its minimal regular model over O_K. More generally, one can even give a sensible definition of the relative discriminant of an arithmetic surface in this way. Since we are only dealing with curves with semi-stable reduction over K, we do not give a precise definition, but rather refer the interested reader to Saito [54].

1.5. Arakelov invariants of curves over $\overline{\mathbf{Q}}$

The following lemma asserts that Arakelov invariants of curves with semi-stable reduction are "stable".

Lemma 1.5.1. *Let* K *be a number field and let* X_0 *be a smooth projective geometrically connected curve over* K *of positive genus. Assume that the minimal regular model of* X_0 *over* O_K *is semi-stable over* O_K. *Then, for any finite field extension* L/K, *we have*

$$h_{\mathrm{Fal}}\,(X_0/K)\,[L : K] = h_{\mathrm{Fal}}((X_0 \times_K L)/L),$$

$$\Delta(X_0/K)[L:K] = \Delta((X_0 \times_K L)/L),$$
$$e(X_0/K)[L:K] = e((X_0 \times_K L)/L).$$

Proof. This follows from the second, fourth, and sixth assertion of Theorem 1.4.5. $\square$

Remark 1.5.2. Let X be a smooth projective geometrically connected curve over a number field K. One can consider *stable Arakelov invariants of* X. These are defined as follows. Let L/K be a finite field extension such that X_L has semi-stable reduction over O_L. Then the stable Arakelov invariants of X over K are defined as

$$h_{\mathrm{Fal,stable}}(X) = \frac{h_{\mathrm{Fal}}(X_L/L)}{[L:\mathbf{Q}]}, \quad e_{\mathrm{stable}}(X) = \frac{e(X_L/L)}{[L:\mathbf{Q}]},$$
$$\Delta_{\mathrm{stable}}(X) = \frac{\Delta(X_L/L)}{[L:\mathbf{Q}]}.$$

By Lemma 1.5.1, these invariants do not depend on the choice of field extension L/K.

Let $\mathbf{Q} \to \overline{\mathbf{Q}}$ be an algebraic closure of the field of rational numbers $\mathbf{Q}$. Let X be a smooth projective connected curve over $\overline{\mathbf{Q}}$ of positive genus. There exists a number field K, an embedding $K \to \overline{\mathbf{Q}}$ and a model X_0 over K for X, with respect to the embedding $K \to \overline{\mathbf{Q}}$, such that the minimal regular model of X_0 over O_K is semi-stable. This follows from the semi-stable reduction theorem (Theorem 1.4.4). We wish to show that the real numbers

$$h_{\mathrm{Fal,stable}}(X_0), \quad e_{\mathrm{stable}}(X_0), \quad \text{and} \quad \Delta_{\mathrm{stable}}(X_0)$$

are invariants of X over $\overline{\mathbf{Q}}$, i.e., they do not depend on the choice of K, $K \to \overline{\mathbf{Q}}$ and X_0. This boils down to the following lemma.

Lemma 1.5.3. *Let $K/\mathbf{Q}$ be a finite Galois extension with ring of integers O_K. Let $p : \mathcal{X} \to \operatorname{Spec} O_K$ be a semi-stable arithmetic surface. Then, for any g in the Galois group $\operatorname{Gal}(K/\mathbf{Q})$, the equalities*

$$h_{\mathrm{Fal}}(\mathcal{X}) = h_{\mathrm{Fal}}(g\mathcal{X}), \quad e(\mathcal{X}) = e(g\mathcal{X}), \quad \Delta(\mathcal{X}) = \Delta(g\mathcal{X})$$

hold, where $g\mathcal{X}$ is the conjugate of $\mathcal{X}$ with respect to g.

Proof. Since g permutes the finite places of K with the same residue characteristic, it is clear that $\Delta(\mathcal{X}) = \Delta(g\mathcal{X})$. Note that $h_{\mathrm{Fal}}(\mathcal{X}) = h_{\mathrm{Fal}}(g\mathcal{X})$. In fact, we have a cartesian diagram

$$\begin{array}{ccc} g\mathcal{X} & \longrightarrow & \mathcal{X} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ \operatorname{Spec} O_K & \xrightarrow[g]{} & \operatorname{Spec} O_K. \end{array}$$

Note that $g^* \det p_* \omega_{\mathcal{X}/O_K} = \det q_* \omega_{g\mathcal{X}/O_K}$. By the Galois invariance of the Arakelov degree $\widehat{\deg}$, we conclude that

$$h_{\mathrm{Fal}}(\mathcal{X}) = \widehat{\deg} \det p_* \omega_{\mathcal{X}/O_K} = \widehat{\deg}\, g^* \det q_* \omega_{g\mathcal{X}/O_K} = \widehat{\deg} \det q_* \omega_{g\mathcal{X}/O_K}.$$

The latter clearly equals $h_{\mathrm{Fal}}(g\mathcal{X})$. A similar reasoning applies to the self-intersection of the dualizing sheaf $e(\mathcal{X})$. $\square$

We are now ready to define Arakelov invariants of X over $\overline{\mathbf{Q}}$. We define

$$\delta_{\mathrm{Fal}}(X) := \frac{\delta_{\mathrm{Fal}}(X_0/K)}{[K:\mathbf{Q}]}, \quad \|\vartheta\|_{\max}(X) := \|\vartheta\|_{\max}(X_0/K)^{1/[K:\mathbf{Q}]},$$
$$S(X) := S(X_0/K)^{1/[K:\mathbf{Q}]}, \quad R(X) := R(X_0/K)^{1/[K:\mathbf{Q}]}.$$

We will refer to $\delta_{\mathrm{Fal}}(X)$ as the *Faltings delta invariant* of X. We also define

$$h_{\mathrm{Fal}}(X) := h_{\mathrm{Fal,stable}}(X_0), \quad e(X) := e_{\mathrm{stable}}(X_0), \quad \Delta(X) := \Delta_{\mathrm{stable}}(X_0).$$

We will refer to $h_{\mathrm{Fal}}(X)$ as the *Faltings height* of X, to $e(X)$ as the *self-intersection of the dualizing sheaf* of X and to $\Delta(X)$ as the *discriminant* of X.

1.6. The stable Faltings height of an abelian variety

In this section we state two important properties of the Faltings height of a curve over $\overline{\mathbf{Q}}$. Let us be more precise.

Let K be a number field, and let A be a g-dimensional abelian variety over K. Let $\mathcal{A}$ be the Néron model of A over O_K; see [7]. Then we have the locally free O_K-module $\mathrm{Cot}_0(A) := 0^*\Omega_{\mathcal{A}/O_K}$ of rank g, and hence the invertible O_K-module of rank one:

$$\omega_A := \Lambda^g \mathrm{Cot}_0(A).$$

For each complex embedding $\sigma : K \to \mathbf{C}$, we have the scalar product on $\mathbf{C} \otimes_{O_K} \omega_A$ given by

$$(\omega, \eta) = \frac{i}{2}(-1)^{g(g-1)/2} \int_{A_\sigma(\mathbf{C})} \omega \overline{\eta}.$$

The *relative Faltings height* of A over K is then defined to be the Arakelov degree of the metrized line bundle ω_A,

$$h_{\mathrm{Fal}}(A/K) = \widehat{\deg}\, \omega_J.$$

Recall that A has *semi-stable reduction* over O_K if the unipotent rank of each special fibre of $\mathcal{A}$ over O_K equals zero. By the semi-stable reduction theorem for abelian varieties (see [1]), there exists a finite field extension L/K such that A_L has semi-stable reduction over O_L.

Definition 1.6.1. Let L/K be a finite field extension such that A_L has semi-stable reduction over O_L. Then the *stable Faltings height* of A is defined to be

$$h_{\mathrm{Fal,stable}}(A) := \frac{h_{\mathrm{Fal}}(A_L/L)}{[L:K]}.$$

Definition 1.6.2. Let A be an abelian variety over $\overline{\mathbf{Q}}$. Let K be a number field such that the abelian variety A has a model A_0 over K with semi-stable reduction over O_K. Then the *Faltings height* of A is defined as $h_{\mathrm{Fal}}(A) := h_{\mathrm{Fal,stable}}(A_0)$.

To show that these invariants are well-defined one applies arguments similar to those given in the proofs of Lemma 1.5.1 and Lemma 1.5.3. For the sake of completeness, we now state two important properties of the Faltings height.

Theorem 1.6.3. *Let X be a curve over $\overline{\mathbf{Q}}$ of positive genus. Then*

$$h_{\mathrm{Fal}}(X) = h_{\mathrm{Fal}}(\mathrm{Jac}(X)).$$

Proof. See Lemme 3.2.1 of Chapter 1 in [59]. □

The Faltings height has the following Northcott property.

Theorem 1.6.4. (Faltings) *Let C be a real number and let g be an integer. For a number field K, there are only finitely many K-isomorphism classes of g-dimensional principally polarized abelian varieties A over K such that A has semi-stable reduction over O_K and $h_{\mathrm{Fal}} \leq C$.*

Proof. This is shown in [23]. An alternative proof was given by Pazuki in [50]. □

This implies the Northcott property for the Faltings height of curves.

Theorem 1.6.5. *Let C be a real number, and let $g \geq 2$ be an integer. For a number field K, there are only finitely many K-isomorphism classes of smooth projective connected curves X over K of genus g with semi-stable reduction over O_K and $h_{\mathrm{Fal},stable}(X) \leq C$.*

Proof. The Faltings height of X coincides with the Faltings height of its Jacobian J; see Theorem 1.6.3. Moreover, X has semi-stable reduction over O_K if and only if J has semi-stable reduction over O_K; see [18]. Thus, the result follows from Torelli's theorem (and a standard Galois cohomology argument as in Remark 4.1.4). □

1.7. Arakelov height and the Arakelov norm of the Wronskian

The main goal of this thesis is to obtain bounds for the Arakelov invariants defined in Section 1.5. To do this, we introduce the height function on a curve. Let us be more precise.

Let X be a smooth projective connected curve over $\overline{\mathbf{Q}}$ of positive genus. We introduce two functions on $X(\overline{\mathbf{Q}})$: the height and the Arakelov norm of the Wronskian differential. More precisely, let $b \in X(\overline{\mathbf{Q}})$. Let K be a number field, $K \to \overline{\mathbf{Q}}$ an embedding and X_0 a smooth projective geometrically connected curve over K whose minimal regular model $\mathcal{X} \to \operatorname{Spec} O_K$ over O_K is semi-stable such that $X_0 \times_K \overline{\mathbf{Q}}$ is isomorphic to X over $\overline{\mathbf{Q}}$ and b induces a section P of $\mathcal{X}$ over O_K. Then we define the *(canonical Arakelov) height of* b, denoted by $h(b)$, to be

$$h(b) = \frac{\widehat{\deg} P^*\omega_{\mathcal{X}/O_K}}{[K:\mathbf{Q}]} = \frac{(P, \omega_{\mathcal{X}/O_K})}{[K:\mathbf{Q}]}.$$

Note that the height of b is the stable canonical height of a point, in the Arakelov-theoretic sense, with respect to the admissible line bundle $\omega_{\mathcal{X}/O_K}$. That is, let K be a number field, $K \to \overline{\mathbf{Q}}$ an embedding and X_0 a smooth projective geometrically connected curve over K whose minimal regular model $\mathcal{X} \to \operatorname{Spec} O_K$ over O_K is semi-stable such that $X_0 \times_K \overline{\mathbf{Q}}$ is isomorphic to X over $\overline{\mathbf{Q}}$ and b induces an algebraic point b_K of X. If D denotes the Zariski closure of b_K in $\mathcal{X}$, then

$$h(b) = \frac{(D, \omega_{\mathcal{X}/O_K})}{[K:\mathbf{Q}] \deg(D/K)}.$$

The height is a well-defined function, i.e., independent of the choice of K, $K \to \overline{\mathbf{Q}}$ and X_0. To prove this, one can argue as in Section 1.5.

Moreover, we define the *Arakelov norm of the Wronskian differential*

$$\|\mathrm{Wr}\| : X(\overline{\mathbf{Q}}) \to \mathbf{R}_{\geq 0}$$

as

$$\|\mathrm{Wr}\|_{\mathrm{Ar}}(b) = \left(\prod_{\sigma: K \to \mathbf{C}} \|\mathrm{Wr}\|_{\mathrm{Ar}}(b_\sigma) \right)^{1/[K:\mathbf{Q}]}.$$

Example 1.7.1. For the reader's convenience, we collect some explicit formulas for elliptic curves from [16], [24] and [57]. Suppose that $X/\overline{\mathbf{Q}}$ is an elliptic curve. Then $e(X) = 0$ and

$$12 h_{\mathrm{Fal}}(X) = \Delta(X) + \delta_{\mathrm{Fal}}(X) - 4\log(2\pi).$$

One can relate $\Delta(X)$ and $\delta_{\mathrm{Fal}}(X)$ to some classical invariants. In fact, let K, $K \to \overline{\mathbf{Q}}$, $X_0 \to \operatorname{Spec} K$ and $\mathcal{X} \to \operatorname{Spec} O_K$ be as above. Let D be the minimal discriminant of the elliptic curve $X_0 \to \operatorname{Spec} K$ and let $\|\Delta\|(X_{0,\sigma})$ be the modular discriminant of the complex elliptic curve $X_{0,\sigma}$, where $\sigma : O_K \to \mathbf{C}$ is a complex embedding. Then

$$\Delta(X) = \log |N_{K/\mathbf{Q}}(D)|,$$

where $N_{K/\mathbf{Q}}$ is the norm with respect to $K/\mathbf{Q}$. Moreover,

$$[K:\mathbf{Q}]\delta_{\mathrm{Fal}}(X) + [K:\mathbf{Q}] 8 \log(2\pi) = \sum_{\sigma: O_K \to \mathbf{C}} -\log \|\Delta\|(X_{0,\sigma}).$$

Szpiro showed that, for any $b \in X(\overline{\mathbf{Q}})$, we have $12h(b) = \Delta(X)$. In particular, the "height" function on $X(\overline{\mathbf{Q}})$ is constant. Therefore, $h : X(\overline{\mathbf{Q}}) \to \mathbf{R}_{\geq 0}$ is not a "height" function in the usual sense when $g = 1$.

If $g \geq 2$, for any real number A, there exists a point $x \in X(\overline{\mathbf{Q}})$ such that $h(x) \geq A$; see [58, Exposé XI, Section 3.2]. Also, if $g \geq 2$, the canonical Arakelov height function on $X(\overline{\mathbf{Q}})$ has the following Northcott property. For any real number C and integer d, there are only finitely many x in $X(\overline{\mathbf{Q}})$ such that $h(x) \leq C$ and $[\mathbf{Q}(x) : \mathbf{Q}] \leq d$. Faltings showed that, for all x in $X(\overline{\mathbf{Q}})$, the inequality $h(x) \geq 0$ holds; see [24, Theorem 5]. In particular, when $g \geq 2$, the function $h : X(\overline{\mathbf{Q}}) \to \mathbf{R}_{\geq 0}$ is a height function in the usual sense.

Changing the model for X might change the height of a point. Let us show that the height of a point does not become smaller if we take another regular model over O_K.

Lemma 1.7.2. *Let $\mathcal{X}' \to \operatorname{Spec} O_K$ be an arithmetic surface such that the generic fibre $\mathcal{X}'_K$ is isomorphic to $\mathcal{X}_K$. Suppose that $b \in X(\overline{\mathbf{Q}})$ induces a section Q of $\mathcal{X}' \to \operatorname{Spec} O_K$. Then*

$$h(b) \leq \frac{(Q, \omega_{\mathcal{X}'/O_K})}{[K : \mathbf{Q}]}.$$

Proof. By the minimality of $\mathcal{X}$, there is a unique birational morphism $\phi : \mathcal{X}' \to \mathcal{X}$; see [41, Corollary 9.3.24]. By the factorization theorem, this morphism is made up of a finite sequence

$$\mathcal{X}' = \mathcal{X}_n \xrightarrow{\phi_n} \mathcal{X}_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_1} \mathcal{X}_0 = \mathcal{X}$$

of blowing-ups along closed points; see [41, Theorem 9.2.2]. For an integer $i = 1, \ldots, n$, let $E_i \subset \mathcal{X}_i$ denote the exceptional divisor of ϕ_i. Since the line bundles $\omega_{\mathcal{X}_i/O_K}$ and $\phi_i^* \omega_{\mathcal{X}_{i-1}/O_K}$ agree on $\mathcal{X}_i - E_i$, there is an integer a such that

$$\omega_{\mathcal{X}_i/O_K} = \phi_i^* \omega_{\mathcal{X}_{i-1}/O_K} \otimes_{\mathcal{O}_{\mathcal{X}_i}} \mathcal{O}_{\mathcal{X}_i}(aE_i).$$

Applying the adjunction formula, we see that $a = 1$. Since ϕ_i restricts to the identity morphism on the generic fibre, we have a canonical isomorphism of admissible line bundles

$$\omega_{\mathcal{X}_i/O_K} = \phi_i^* \omega_{\mathcal{X}_{i-1}/O_K} \otimes_{\mathcal{O}_{\mathcal{X}_i}} \mathcal{O}_{\mathcal{X}_i}(E_i).$$

Let Q_i denote the section of $\mathcal{X}_i$ over O_K induced by $b \in X(\overline{\mathbf{Q}})$. Then

$$\begin{aligned} (Q_i, \omega_{\mathcal{X}_i/O_K}) &= (Q_i, \phi_i^* \omega_{\mathcal{X}_{i-1}/O_K}) + (Q_i, E_i) \geq (Q_i, \phi_i^* \omega_{\mathcal{X}_{i-1}/O_K}) \\ &= (Q_{i-1}, \omega_{\mathcal{X}_{i-1}/O_K}), \end{aligned}$$

where we used the projection formula in the last equality. Therefore,

$$(Q, \omega_{\mathcal{X}'/O_K}) = (Q_n, \omega_{\mathcal{X}_n/O_K}) \geq (Q_0, \omega_{\mathcal{X}_0/O_K}) = (P, \omega_{\mathcal{X}/O_K}).$$

Since $(P, \omega_{\mathcal{X}/O_K}) = h(b)[K : \mathbf{Q}]$, this concludes the proof. □

1.8. A lower bound for the height of a non-Weierstrass point

We follow [15] in this section.

Proposition 1.8.1. *Let X be a smooth projective connected curve over $\overline{\mathbf{Q}}$ of genus $g \geq 1$. Then, for any non-Weierstrass point b in $X(\overline{\mathbf{Q}})$,*

$$\frac{1}{2}g(g+1)h(b) + \log \|\mathrm{Wr}\|_{\mathrm{Ar}}(b) \geq h_{\mathrm{Fal}}(X).$$

Proof. This follows from [15, Proposition 5.9]. Let us explain this. Let K be a number field such that X has a model X_0 over K with semi-stable reduction over O_K and the property that b is rational over K. Then, if $p : \mathcal{X} \to \operatorname{Spec} O_K$ is the minimal regular (semi-stable) model of X_0 over O_K, by [15, Proposition 5.9], the real number $\frac{1}{2}g(g+1)(P, \omega_{\mathcal{X}/O_K})$ equals

$$h_{\mathrm{Fal}}(\mathcal{X}) - \sum_{\sigma: K \to \mathbf{C}} \log \|\mathrm{Wr}\|_{\mathrm{Ar}, X_\sigma}(b_\sigma) + \log\left(\# R^1 p_* \mathcal{O}_{\mathcal{X}}(gP)\right),$$

where we let P denote the section of $p : \mathcal{X} \to \operatorname{Spec} O_K$ induced by b. Since

$$\log\left(\# R^1 p_* \mathcal{O}_{\mathcal{X}}(gP)\right) \geq 0,$$

the inequality

$$\frac{1}{2}g(g+1)(P, \omega_{\mathcal{X}/O_K}) \geq h_{\mathrm{Fal}}(\mathcal{X}) - \sum_{\sigma: K \to \mathbf{C}} \log \|\mathrm{Wr}\|_{\mathrm{Ar}, X_\sigma}(b_\sigma)$$

holds. Dividing both sides by $[K : \mathbf{Q}]$ gives the sought inequality. In fact, by definition,

$$h(b) = \frac{(P, \omega_{\mathcal{X}/O_K})}{[K : \mathbf{Q}]}, \quad h_{\mathrm{Fal}}(X) = \frac{h_{\mathrm{Fal}}(\mathcal{X})}{[K : \mathbf{Q}]},$$

and

$$\log \|\mathrm{Wr}\|_{\mathrm{Ar}}(b) = \frac{1}{[K : \mathbf{Q}]} \sum_{\sigma: K \to \mathbf{C}} \log \|\mathrm{Wr}\|_{\mathrm{Ar}, X_\sigma}(b_\sigma).$$ □

1.9. The Belyi degree of a curve

We finish this chapter with a discussion of the Belyi degree of a smooth projective connected curve over $\overline{\mathbf{Q}}$.

Theorem 1.9.1. *Let X be a smooth projective connected curve over $\mathbf{C}$. Then the following assertions are equivalent.*

1. *The curve X can be defined over a number field.*
2. *There exists a finite morphism $X \to \mathbf{P}^1_{\mathbf{C}}$ ramified over precisely three points.*

Proof. Weil (and later Grothendieck) showed that "2 implies 1"; see [27]. In [5] Belyi proved that "1 implies 2". □

Example 1.9.2. Let $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ be a finite index subgroup. Then the compactification X_Γ of the Riemann surface $\Gamma\backslash\mathbf{H}$ (obtained by adding cusps) can be defined over a number field. This follows from the implication (2) $\implies$ (1) of Theorem 1.9.1. In fact, the morphism $X_\Gamma \to X(1) \cong \mathbf{P}^1_{\mathbf{C}}$ of degree at most $[\mathrm{SL}_2(\mathbf{Z}) : \Gamma]$ is ramified over precisely three points if $g(X_\Gamma) \geq 1$. (The isomorphism $X(1) \cong \mathbf{P}^1(\mathbf{C})$ is given by the j-invariant.)

Example 1.9.3. Let $n \geq 4$ be an integer. Let $F(n)$ be the curve defined by the equation $x^n + y^n = z^n$ in $\mathbf{P}^2_{\mathbf{C}}$. We call $F(n)$ the Fermat curve of degree n. The morphism from $F(n)$ to $\mathbf{P}^1_{\overline{\mathbf{Q}}}$ given by $(x : y : z) \mapsto (x^n : z^n)$ is ramified over precisely three points. We note that this finite morphism is of degree n^2.

Definition 1.9.4. Let X be a smooth projective connected curve over $\mathbf{C}$ which can be defined over a number field. Then the Belyi degree of X, denoted by $\deg_B(X)$, is defined as the minimal degree of a finite morphism $X \to \mathbf{P}^1_{\mathbf{C}}$ ramified over precisely three points.

Remark 1.9.5. Let U over $\overline{\mathbf{Q}}$ be a smooth quasi-projective connected variety over $\overline{\mathbf{Q}}$. Then base-change from $\overline{\mathbf{Q}}$ to $\mathbf{C}$ (with respect to any embedding $\overline{\mathbf{Q}} \to \mathbf{C}$) induces an equivalence of categories from the category of finite étale covers of U to the category of finite étale covers of $U_{\mathbf{C}}$; see [27]

Definition 1.9.6. Let X be a smooth projective connected curve over $\overline{\mathbf{Q}}$. Then the Belyi degree of X, denoted by $\deg_B(X)$, is defined as the minimal degree of a finite morphism $X \to \mathbf{P}^1_{\overline{\mathbf{Q}}}$ ramified over precisely three points. (Note that such a morphism always exists by Remark 1.9.5.)

Definition 1.9.7. Let X be a curve over a number field K. Let $K \to \mathbf{C}$ be a complex embedding. We define the Belyi degree $\deg_B(X)$ of X to be the Belyi degree of $X_{\mathbf{C}}$. This real number is well-defined, i.e., it does not depend on the choice of the embedding $K \to \mathbf{C}$.

Example 1.9.8. The Belyi degree of the curve X_Γ is bounded from above by the index of Γ in $\mathrm{SL}_2(\mathbf{Z})$.

Example 1.9.9. For all $n \geq 1$, the Belyi degree of the Fermat curve $F(n)$ is bounded by n^2.

Lemma 1.9.10. *For X a smooth projective connected over $\overline{\mathbf{Q}}$ of genus g, we have $2g+1 \leq \deg_B(X)$.*

Proof. Let $\pi : X \to \mathbf{P}^1_{\overline{\mathbf{Q}}}$ be ramified over precisely three points. By Riemann-Hurwitz, the equality $2g - 2 = -2\deg\pi + \deg R$ holds, where R is the ramification divisor of $\pi : X \to \mathbf{P}^1_{\overline{\mathbf{Q}}}$. The lemma follows from the inequality $\deg R \leq 3\deg\pi - 3$. □

Example 1.9.11. The Belyi degree of the genus g curve $y^2 + y = x^{2g+1}$ equals $2g + 1$. In fact, the projection onto y is a Belyi cover of degree $2g + 1$. In particular, the inequality of Lemma 1.9.10 is sharp.

Proposition 1.9.12. *Let C be a real number. The set of $\overline{\mathbf{Q}}$-isomorphism classes of smooth projective connected curves X such that $\deg_B(X) \leq C$ is finite.*

Proof. The fundamental group of the Riemann sphere minus three points is finitely generated.

□

CHAPTER 2

Polynomial bounds for Arakelov invariants of Belyi curves

This chapter forms the technical heart of this thesis. Most of the results of this chapter also appear in our article [30].

2.1. Main result

We prove that stable Arakelov invariants of a curve over a number field are polynomial in the Belyi degree. We use our results to give algorithmic, geometric and Diophantine applications in the following two chapters.

Let X be a smooth projective connected curve over $\overline{\mathbf{Q}}$ of genus g. In [5] Belyi proved that there exists a finite morphism $X \to \mathbf{P}^1_{\overline{\mathbf{Q}}}$ ramified over at most three points. Let $\deg_B(X)$ denote the Belyi degree of X (introduced in Section 1.9). Since the topological fundamental group of the projective line $\mathbf{P}^1(\mathbf{C})$ minus three points is finitely generated, the set of $\overline{\mathbf{Q}}$-isomorphism classes of curves with bounded Belyi degree is finite; see Proposition 1.9.12. In particular, the "height" of X is bounded in terms of $\deg_B(X)$.

We prove that, if $g \geq 1$, the Faltings height $h_{\mathrm{Fal}}(X)$, the Faltings delta invariant $\delta_{\mathrm{Fal}}(X)$, the discriminant $\Delta(X)$ and the self-intersection of the dualizing sheaf $e(X)$ are bounded by an explicitly given polynomial in $\deg_B(X)$.

Theorem 2.1.1. *For any smooth projective connected curve X over $\overline{\mathbf{Q}}$ of genus $g \geq 1$,*

$$\begin{array}{ccccc} -\log(2\pi)g & \leq & h_{\mathrm{Fal}}(X) & \leq & 13 \cdot 10^6 g \deg_B(X)^5 \\ 0 & \leq & e(X) & \leq & 3 \cdot 10^7 (g-1) \deg_B(X)^5 \\ 0 & \leq & \Delta(X) & \leq & 5 \cdot 10^8 g^2 \deg_B(X)^5 \\ -10^8 g^2 \deg_B(X)^5 & \leq & \delta_{\mathrm{Fal}}(X) & \leq & 2 \cdot 10^8 g \deg_B(X)^5. \end{array}$$

We were first led to investigate this problem by work of Edixhoven, de Jong and Schepers on covers of complex algebraic surfaces with fixed branch locus; see [22]. They conjectured an arithmetic analogue ([22, Conjecture 5.1]) of their main theorem (Theorem 1.1 in *loc. cit.*). We use our results to prove their conjecture; see Section 3.3 for a more precise statement.

Outline of proof

To prove Theorem 2.1.1 we will use Arakelov theory for curves defined over a number field K. To apply Arakelov theory in this context, we will work with *arithmetic surfaces* associated to such curves. We refer the reader to Section 1.2 for precise definitions.

Firstly, we show that, to prove Theorem 2.1.1, it suffices to bound the canonical height of some non-Weierstrass point and the Arakelov norm of the Wronskian differential at this point; see Theorem 2.2.1 for a precise statement.

In Section 2.3 we have gathered all the necessary analytic results. We estimate Arakelov-Green functions and Arakelov norms of Wronskian differentials on finite étale covers of the modular curve $Y(2)$ in Theorem 2.3.12 and Proposition 2.3.13, respectively. In our proof we use an explicit version of a result of Merkl on the Arakelov-Green function; see Theorem 2.3.2. This version of Merkl's theorem was obtained by Peter Bruin in his master's thesis ([9]). The proof of this version of Merkl's theorem is reproduced in the appendix of [30] by Peter Bruin.

In Section 2.5.2 we prove the existence of a non-Weierstrass point on X of bounded height; see Theorem 2.5.4. The proof of Theorem 2.5.4 relies on our bounds for Arakelov-Green functions (Theorem 2.3.12), the existence of a "wild" model (Theorem 2.4.9) and a generalization of Dedekind's discriminant conjecture for discrete valuation rings of characteristic zero (Proposition 2.4.1) which we attribute to H.W. Lenstra jr.

A precise combination of the above results constitutes the proof of Theorem 2.1.1 given in Section 2.5.3.

2.2. Reduction to bounding the Arakelov height of a point

In this section we prove bounds for Arakelov invariants of curves in the height of a non-Weierstrass point and the Arakelov norm of the Wronskian differential in this point.

Theorem 2.2.1. *Let X be a smooth projective connected curve over $\overline{\mathbf{Q}}$ of genus $g \geq 1$. Let $b \in X(\overline{\mathbf{Q}})$. Then*

$$\begin{aligned} e(X) &\leq 4g(g-1)h(b), \\ \delta_{\mathrm{Fal}}(X) &\geq -90g^3 - 4g(2g-1)(g+1)h(b). \end{aligned}$$

Suppose that b is not a Weierstrass point. Then

$$\begin{aligned} h_{\mathrm{Fal}}(X) &\leq \tfrac{1}{2}g(g+1)h(b)+\log\|\mathrm{Wr}\|_{\mathrm{Ar}}(b), \\ \delta_{\mathrm{Fal}}(X) &\leq 6g(g+1)h(b)+12\log\|\mathrm{Wr}\|_{\mathrm{Ar}}(b)+4g\log(2\pi), \\ \Delta(X) &\leq 2g(g+1)(4g+1)h(b)+12\log\|\mathrm{Wr}\|_{\mathrm{Ar}}(b)+93g^3. \end{aligned}$$

This theorem is essential to the proof of Theorem 2.1.1 given in Section 2.5.2. We give a proof of Theorem 2.2.1 at the end of this section.

Lemma 2.2.2. *For a smooth projective connected curve X over $\overline{\mathbf{Q}}$ of genus $g \geq 1$,*

$$\log\|\vartheta\|_{\max}(X) \leq \frac{g}{4}\log\max(1,h_{\mathrm{Fal}}(X))+(4g^3+5g+1)\log(2).$$

Proof. We kindly thank R. de Jong for sharing this proof with us. We follow the idea of [26, Section 2.3.2], see also [14, Appendice]. Let $\mathcal{F}_g$ be the Siegel fundamental domain of dimension g in the Siegel upper half-space $\mathbf{H}_g$, i.e., the space of complex $(g\times g)$-matrices τ in $\mathbf{H}_g$ such that the following properties are satisfied. Firstly, for every element u_{ij} of $u=\Re(\tau)$, we have $|u_{ij}|\leq 1/2$. Secondly, for every γ in $\mathrm{Sp}(2g,\mathbf{Z})$, we have $\det\Im(\gamma\cdot\tau)\leq\det\Im(\tau)$, and finally, $\Im(\tau)$ is Minkowski-reduced, i.e., for all $\xi=(\xi_1,\ldots,\xi_g)\in\mathbf{Z}^g$ and for all i such that $\xi_i,\ldots,\xi_g$ are non-zero, we have $\xi\Im(\tau)^t\xi\geq(\Im(\tau))_{ii}$ and, for all $1\leq i\leq g-1$ we have $(\Im(\tau))_{i,i+1}\geq 0$. One can show that $\mathcal{F}_g$ contains a representative of each $\mathrm{Sp}(2g,\mathbf{Z})$-orbit in $\mathbf{H}_g$.

Let K be a number field such that X has a model X_K over K. For every embedding $\sigma: K\to\mathbf{C}$, let τ_σ be an element of $\mathcal{F}_g$ such that

$$\mathrm{Jac}(X_{K,\sigma})\cong\mathbf{C}^g/(\tau_\sigma\mathbf{Z}^g+\mathbf{Z}^g)$$

as principally polarized abelian varieties, the matrix of the Riemann form induced by the polarization of $\mathrm{Jac}(X_{K,\sigma})$ being $\Im(\tau_\sigma)^{-1}$ on the canonical basis of $\mathbf{C}^g$. By a result of Bost (see [26, Lemme 2.12] or [50]), we have

$$\frac{\sum_{\sigma:K\to\mathbf{C}}\log\det(\Im(\tau_\sigma))}{[K:\mathbf{Q}]} \leq g\log\max(1,h_{\mathrm{Fal}}(X))+(2g^3+2)\log(2).$$

Here we used that $h_{\mathrm{Fal}}(X)=h_{\mathrm{Fal}}(\mathrm{Jac}(X))$; see Theorem 1.6.3. Now, let $\vartheta(z;\tau)$ be the Riemann theta function as in Section 1.1, where τ is in $\mathcal{F}_g$ and $z=x+iy$ is in $\mathbf{C}^g$ with $x,y\in\mathbf{R}^g$. Combining the latter inequality with the upper bound

$$\exp(-\pi{}^ty(\Im(\tau))^{-1}y)|\vartheta(z;\tau)| \leq 2^{3g^3+5g} \tag{2.2.1}$$

implies the result. Let us prove (2.2.1). Note that, if we write

$$y=\Im(z)=(\Im(\tau))\cdot b$$

for b in $\mathbf{R}^g$,

$$\exp(-\pi {}^t y(\Im(\tau))^{-1}y)|\vartheta(z;\tau)| \leq \sum_{n\in\mathbf{Z}^g} \exp(-\pi {}^t(n+b)(\Im(\tau))(n+b)).$$

Since $\Im(\tau)$ is Minkowski reduced, we have

$${}^t m\Im(\tau)m \geq c(g)\sum_{i=1}^g m_i^2(\Im(\tau))_{ii}$$

for all m in $\mathbf{R}^g$. Here $c(g) = \left(\frac{4}{g^3}\right)^{g-1}\left(\frac{3}{4}\right)^{g(g-1)/2}$. Also, $(\Im(\tau))_{ii} \geq \sqrt{3}/2$ for all $i = 1,\ldots,g$ (cf. [29, Chapter V.4] for these facts). For $i = 1,\ldots,g$, we define

$$B_i := \pi c(g)(n_i+b_i)^2(\Im(\tau))_{ii}.$$

Then, we deduce that

$$\begin{aligned}\sum_{n\in\mathbf{Z}^g}\exp(-\pi {}^t(n+b)(\Im(\tau))(n+b)) &\leq \sum_{n\in\mathbf{Z}^g}\exp\left(-\sum_{i=1}^g B_i\right)\\ &\leq \prod_{i=1}^g\sum_{n_i\in\mathbf{Z}}\exp(-B_i)\end{aligned}$$

Finally, we note that the latter expression is at most

$$\prod_{i=1}^g \frac{2}{1-\exp(-\pi c(g)(\Im(\tau))_{ii})} \leq 2^g\left(1+\frac{2}{\pi\sqrt{3}c(g)}\right)^g.$$

This proves (2.2.1). □

Lemma 2.2.3. *Let $a \in \mathbf{R}_{>0}$ and $b \in \mathbf{R}_{\leq 1}$. Then, for all real numbers $x \geq b$,*

$$x - a\log\max(1,x) = \frac{1}{2}x + \frac{1}{2}(x - 2a\log\max(1,x)),$$

and

$$\frac{1}{2}x + \frac{1}{2}(x-2a\log\max(1,x)) \geq \frac{1}{2}x + \min(\frac{1}{2}b, a - a\log(2a)).$$

Proof. It suffices to prove that $x - 2a\log\max(1,x) \geq \min(b, 2a-2a\log(2a))$ for all $x \geq b$. To prove this, let $x \geq b$. Then, if $2a \leq 1$, we have

$$x - 2a\log\max(1,x) \geq b \geq \min(b, 2a - 2a\log(2a)).$$

(To prove that $x - 2a\log\max(1,x) \geq b$, we may assume that $x \geq 1$. It is easy to show that $x - 2a\log x$ is a non-decreasing function for $x \geq 1$. Therefore, for all $x \geq 1$, we conclude that $x - 2a\log x \geq 1 \geq b$.) If $2a > 1$, the function $x - 2a\log(x)$ attains its minimum value at $x = 2a$ on the interval $[1,\infty)$. This concludes the proof. □

Lemma 2.2.4. (Bost) *Let X be a smooth projective connected curve over $\overline{\mathbf{Q}}$ of genus $g \geq 1$. Then*

$$h_{\mathrm{Fal}}(X) \geq -\log(2\pi)g.$$

Proof. See [25, Corollaire 8.4]. (Note that the Faltings height $h(X)$ utilized by Bost, Gaudron and Rémond is bigger than $h_{\mathrm{Fal}}(X)$ due to a difference in normalization. In fact, we have $h(X) = h_{\mathrm{Fal}}(X) + g\log(\sqrt{\pi})$. In particular, the slightly stronger lower bound

$$h_{\mathrm{Fal}}(X) \geq -\log(\sqrt{2}\pi)g$$

holds.) □

Lemma 2.2.5. *Let X be a smooth projective connected curve over $\overline{\mathbf{Q}}$ of genus $g \geq 1$. Then*

$$\log S(X) + h_{\mathrm{Fal}}(X)$$

is at least

$$\frac{h_{\mathrm{Fal}}(X)}{2} - (4g^3 + 5g + 1)\log(2) + \min\left(\frac{-g\log(2\pi)}{2}, \frac{g}{4} - \frac{g}{4}\log\left(\frac{g}{2}\right)\right).$$

Proof. By the explicit formula (1.1.1) for $S(X)$ and our bounds on theta functions (Lemma 2.2.2),

$$\log S(X) + h_{\mathrm{Fal}}(X)$$

is at least

$$-\frac{g}{4}\log\max(1, h_{\mathrm{Fal}}(X)) - (4g^3 + 5g + 1)\log(2) + h_{\mathrm{Fal}}(X).$$

Since $h_{\mathrm{Fal}}(X) \geq -g\log(2\pi)$, the statement follows from Lemma 2.2.3 (with $x = h_{\mathrm{Fal}}(X)$, $a = g/4$ and $b = -g\log(2\pi)$). □

Lemma 2.2.6. *Let X be a smooth projective connected curve of genus $g \geq 2$ over $\overline{\mathbf{Q}}$. Then*

$$\frac{(2g-1)(g+1)}{8(g-1)}e(X) + \frac{1}{8}\delta_{\mathrm{Fal}}(X) \geq \log S(X) + h_{\mathrm{Fal}}(X).$$

Proof. By [15, Proposition 5.6],

$$e(X) \geq \frac{8(g-1)}{(g+1)(2g-1)}\left(\log R(X) + h_{\mathrm{Fal}}(X)\right).$$

Note that $\log R(X) = \log S(X) - \delta_{\mathrm{Fal}}(X)/8$; see (1.1.2). This implies the inequality. □

Lemma 2.2.7. (Noether formula) *Let X be a smooth projective connected curve over $\overline{\mathbf{Q}}$ of genus $g \geq 1$. Then*

$$12h_{\mathrm{Fal}}(X) = e(X) + \Delta(X) + \delta_{\mathrm{Fal}}(X) - 4g\log(2\pi).$$

Proof. This follows from [24, Theorem 6] and [47, Théorème 2.2]. □

Proposition 2.2.8. *Let X be a smooth projective connected curve of genus $g \geq 2$ over $\overline{\mathbf{Q}}$. Then*

$$\begin{array}{rcl} h_{\mathrm{Fal}}(X) & \leq & \frac{(2g-1)(g+1)}{4(g-1)} e(X) + \frac{1}{4}\delta_{\mathrm{Fal}}(X) + 20g^3 \\ -g\log(2\pi) & \leq & \frac{(2g-1)(g+1)}{4(g-1)} e(X) + \frac{1}{4}\delta_{\mathrm{Fal}}(X) + 20g^3 \\ \Delta(X) & \leq & \frac{3(2g-1)(g+1)}{g-1} e(X) + 2\delta_{\mathrm{Fal}}(X) + 248g^3. \end{array}$$

Proof. Firstly, by Lemma 2.2.6,

$$\frac{(2g-1)(g+1)}{8(g-1)} e(X) + \frac{1}{8}\delta_{\mathrm{Fal}}(X) \geq \log S(X) + h_{\mathrm{Fal}}(X).$$

To obtain the upper bound for $h_{\mathrm{Fal}}(X)$, we proceed as follows. Write

$$s := \log S(X) + h_{\mathrm{Fal}}(X).$$

By Lemma 2.2.5,

$$s \geq \frac{1}{2} h_{\mathrm{Fal}}(X) - (4g^3 + 5g + 1)\log(2) + \min\left(-\frac{g}{2}\log(2\pi), \frac{g}{4} - \frac{g}{4}\log\left(\frac{g}{2}\right)\right).$$

From these two inequalities, we deduce that $\frac{1}{2}h_{\mathrm{Fal}}(X)$ is at most

$$\frac{(2g-1)(g+1)}{8(g-1)} e(X) + \frac{\delta_{\mathrm{Fal}}(X)}{8} + (4g^3 + 5g + 1)\log(2) +$$

$$+ \max\left(\frac{g}{2}\log(2\pi), \frac{g}{4}\log\left(\frac{g}{2}\right) - \frac{g}{4}\right).$$

Finally, it is straightforward to verify the inequality

$$(4g^3 + 5g + 1)\log(2) + \max\left(\frac{g}{2}\log(2\pi), \frac{g}{4}\log\left(\frac{g}{2}\right) - \frac{g}{4}\right) \leq 10g^3.$$

This concludes the proof of the upper bound for $h_{\mathrm{Fal}}(X)$.

The second inequality follows from the first inequality of the proposition and the lower bound $h_{\mathrm{Fal}}(X) \geq -g\log(2\pi)$ of Bost (Lemma 2.2.4).

Finally, to obtain the upper bound of the proposition for the discriminant of X, we eliminate the Faltings height of X in the first inequality using the Noether formula and obtain that

$$\Delta(X) + e(X) + \delta_{\mathrm{Fal}}(X) - 4g\log(2\pi)$$

is at most

$$\frac{3(2g-1)(g+1)}{(g-1)} e(X) + 3\delta_{\mathrm{Fal}}(X) + 240g^3.$$

In [24, Theorem 5] Faltings showed that $e(X) \geq 0$. Therefore, we conclude that

$$\Delta(X) \leq \frac{3(2g-1)(g+1)}{(g-1)} e(X) + 2\delta_{\mathrm{Fal}}(X) + (240 + 4\log(2\pi))g^3.$$ □

We are now ready to prove Theorem 2.2.1.

Proof of Theorem 2.2.1. The proof is straightforward. The upper bound

$$e(X) \leq 4g(g-1)h(b)$$

is well-known; see [24, Theorem 5].

Let us prove the lower bound for $\delta_{\mathrm{Fal}}(X)$. If $g \geq 2$, the lower bound for $\delta_{\mathrm{Fal}}(X)$ can be deduced from the second inequality of Proposition 2.2.8 and the upper bound $e(X) \leq 4g(g-1)h(b)$. When $g = 1$, we can easily compute an explicit lower bound for $\delta_{\mathrm{Fal}}(X)$. For instance, it not hard to show that $\delta_{\mathrm{Fal}}(X) \geq -8\log(2\pi)$ (using the explicit description of $\delta_{\mathrm{Fal}}(X)$ as in Remark 1.7.1).

From now on, we suppose that b is a non-Weierstrass point. The upper bound

$$h_{\mathrm{Fal}}(X) \leq \frac{1}{2}g(g+1)h(b) + \log \|\mathrm{Wr}\|_{\mathrm{Ar}}(b)$$

is Proposition 1.8.1.

We deduce the upper bound

$$\delta_{\mathrm{Fal}}(X) \leq 6g(g+1)h(b) + 12\log \|\mathrm{Wr}\|_{\mathrm{Ar}}(b) + 4g\log(2\pi)$$

as follows. Since $e(X) \geq 0$ and $\Delta(X) \geq 0$, the Noether formula implies that

$$\delta_{\mathrm{Fal}}(X) \leq 12h_{\mathrm{Fal}}(X) + 4g\log(2\pi).$$

Thus, the upper bound for $\delta_{\mathrm{Fal}}(X)$ follows from the upper bound for $h_{\mathrm{Fal}}(X)$.

Finally, the upper bound

$$\Delta(X) \leq 2g(g+1)(4g+1)h(b) + 12\log \|\mathrm{Wr}\|_{\mathrm{Ar}}(b) + 93g^3$$

follows from the inequality $\Delta(X) \leq 12h_{\mathrm{Fal}}(X) - \delta_{\mathrm{Fal}}(X) + 4g\log(2\pi)$ and the preceding bounds. (One could also use the last inequality of Proposition 2.2.8 to obtain a similar result.) □

2.3. Analytic part

Our aim is to give explicit bounds for the Arakelov-Green function on a Belyi cover of $X(2)$ in this section. Such bounds have been obtained for certain Belyi covers using spectral methods in [33]. The results in *loc. cit.* do not apply to our situation since the smallest positive eigenvalue of the Laplacian can go to zero in a tower of Belyi covers; see [43, Theorem 4].

Instead, we use a theorem of Merkl to prove explicit bounds for the Arakelov-Green function on a Belyi cover in Theorem 2.3.12. More precisely, we construct a "Merkl atlas" for an arbitrary Belyi cover. Our construction uses an explicit version of a result of Jorgenson and Kramer ([32]) on the Arakelov $(1,1)$-form due to Bruin.

We use our results to estimate the Arakelov norm of the Wronskian differential in Proposition 2.3.13.

2.3.1. Merkl's theorem

Let X be a compact connected Riemann surface of positive genus and recall that μ denotes the Arakelov $(1,1)$-form on X.

Definition 2.3.1. A *Merkl atlas* for X is a quadruple

$$(\{(U_j, z_j)\}_{j=1}^n, r_1, M, c_1),$$

where $\{(U_j, z_j)\}_{j=1}^n$ is a finite atlas for X, $\frac{1}{2} < r_1 < 1$, $M \geq 1$ and $c_1 > 0$ are real numbers such that the following properties are satisfied.

1. Each $z_j U_j$ is the open unit disc.
2. The open sets $U_j^{r_1} := \{x \in U_j : |z_j(x)| < r_1\}$ with $1 \leq j \leq n$ cover X.
3. For all $1 \leq j, j' \leq n$, the function $|dz_j/dz_{j'}|$ on $U_j \cap U_{j'}$ is bounded from above by M.
4. For $1 \leq j \leq n$, write $\mu_{\mathrm{Ar}} = iF_j dz_j \wedge d\overline{z_j}$ on U_j. Then $0 \leq F_j(x) \leq c_1$ for all $x \in U_j$.

Given a Merkl atlas $(\{(U_j, z_j)\}_{j=1}^n, r_1, M, c_1)$ for X, the following result provides explicit bounds for Arakelov-Green functions in n, r_1, M and c_1.

Theorem 2.3.2 (Merkl)**.** *Let* $(\{(U_j, z_j)\}_{j=1}^n, r_1, M, c_1)$ *be a Merkl atlas for* X*. Then*

$$\sup_{X \times X \setminus \Delta} \mathrm{gr}_X \leq \frac{330n}{(1-r_1)^{3/2}} \log \frac{1}{1-r_1} + 13.2nc_1 + (n-1)\log M.$$

Furthermore, for every index j *and all* $x \neq y \in U_j^{r_1}$*, we have that*

$$|\,\mathrm{gr}_X(x,y) - \log|z_j(x) - z_j(y)||$$

is at most

$$\frac{330n}{(1-r_1)^{3/2}} \log \frac{1}{1-r_1} + 13.2nc_1 + (n-1)\log M.$$

Proof. Merkl proved this theorem without explicit constants and without the dependence on r_1 in [45]. A proof of the theorem in a more explicit form was given by P. Bruin in his master's thesis; see [9]. This proof is reproduced, with minor modifications, in the appendix of [30]. □

2.3.2. An atlas for a Belyi cover of $X(2)$

Let $\mathbf{H}$ denote the complex upper half-plane. Recall that $\mathrm{SL}_2(\mathbf{R})$ acts on $\mathbf{H}$ via Möbius transformations. Let $\Gamma(2)$ denote the subgroup of $\mathrm{SL}_2(\mathbf{Z})$ defined as

$$\Gamma(2) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z}) : a \equiv d \equiv 1 \mod 2 \text{ and } b \equiv c \equiv 0 \mod 2\}.$$

The Riemann surface $Y(2) = \Gamma(2)\backslash\mathbf{H}$ is not compact. Let $X(2)$ be the compactification of the Riemann surface $Y(2) = \Gamma(2)\backslash\mathbf{H}$ obtained by adding the cusps 0, 1 and ∞. Note that $X(2)$ is known as the *modular curve associated to the congruence subgroup* $\Gamma(2)$ *of* $\mathrm{SL}_2(\mathbf{Z})$. The modular lambda function $\lambda : \mathbf{H} \to \mathbf{C}$ induces an analytic isomorphism $\lambda : X(2) \to \mathbf{P}^1(\mathbf{C})$; see Section 2.5.1 for details. In particular, the genus of $X(2)$ is zero. For a cusp $\kappa \in \{0, 1, \infty\}$, we fix an element γ_κ in $\mathrm{SL}_2(\mathbf{Z})$ such that $\gamma_\kappa(\kappa) = \infty$.

We construct an atlas for the compact connected Riemann surface $X(2)$. Let $\dot{B}_\infty$ be the open subset given by the image of the strip

$$\dot{S}_\infty := \left\{ x + iy : -1 \leq x < 1, y > \frac{1}{2} \right\} \subset \mathbf{H}$$

in $Y(2)$ under the quotient map $\mathbf{H} \longrightarrow \Gamma(2)\backslash\mathbf{H}$ defined by $\tau \mapsto \Gamma(2)\tau$. The quotient map $\mathbf{H} \longrightarrow \Gamma(2)\backslash\mathbf{H}$ induces a bijection from this strip to $\dot{B}_\infty$. More precisely, suppose that τ and τ' in $\dot{S}_\infty$ lie in the same orbit under the action of $\Gamma(2)$. Then, there exists an element

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$$

such that $\gamma\tau = \tau'$. If $c \neq 0$, by definition, c is a non-zero integral multiple of 2. Thus, $c^2 \geq 4$. Therefore,

$$\frac{1}{2} < \Im\tau' = \frac{\Im\tau}{|c\tau + d|^2} \leq \frac{1}{4\Im\tau} < \frac{1}{2}.$$

This is clearly impossible. Thus, $c = 0$ and $\tau' = \tau \pm b$. By definition, $b = 2k$ for some integer k. Since τ and τ' lie in the above strip, we conclude that $b = 0$. Thus $\tau = \tau'$.

Consider the morphism $z_\infty : \mathbf{H} \longrightarrow \mathbf{C}$ given by $\tau \mapsto \exp(\pi i\tau + \frac{\pi}{2})$. The image of the strip $\dot{S}_\infty$ under z_∞ in $\mathbf{C}$ is the punctured open unit disc $\dot{B}(0,1)$. Now, for any τ and τ' in the strip $\dot{S}_\infty$, the equality $z_\infty(\tau) = z_\infty(\tau')$ holds if and only if $\tau' = \tau \pm 2k$ for some integer k. But then $k = 0$ and $\tau = \tau'$. We conclude that z_∞ factors injectively through $\dot{B}_\infty$. Let $z_\infty : B_\infty \longrightarrow B(0,1)$ denote, by abuse of notation, the induced chart at ∞, where $B_\infty := \dot{B}_\infty \cup \{\infty\}$ and $B(0,1)$ is the open unit disc in $\mathbf{C}$. We translate our neighbourhood B_∞ at ∞ to a neighborhood for κ, where κ is a cusp of $X(2)$. More precisely, for any τ in $\mathbf{H}$, define $z_\kappa(\tau) = \exp(\pi i\gamma_k^{-1}\tau + \pi/2)$. Let $\dot{B}_\kappa$ be the image of $\dot{S}_\infty$ under the map $\mathbf{H} \longrightarrow Y(2)$ given by

$$\tau \mapsto \Gamma(2)\gamma_\kappa\tau.$$

We define $B_\kappa = \dot{B}_\kappa \cup \{\kappa\}$. We let $z_\kappa : B_\kappa \to B(0,1)$ denote the induced chart (by abuse of notation).

Since the open subsets B_κ cover $X(2)$, we have constructed an atlas $\{(B_\kappa, z_\kappa)\}_\kappa$ for $X(2)$, where κ runs through the cusps 0, 1 and ∞.

Definition 2.3.3. A *Belyi cover* of $X(2)$ is a morphism of compact connected Riemann surfaces $Y \longrightarrow X(2)$ which is unramified over $Y(2)$. The points of Y not lying over $Y(2)$ are called *cusps*.

Lemma 2.3.4. *Let $\pi : Y \longrightarrow X(2)$ be a Belyi cover with Y of genus g. Then, $g \leq \deg \pi$.*

Proof. This follows from Lemma 1.9.10. □

Let $\pi : Y \longrightarrow X(2)$ be a Belyi cover. We are going to "lift" the atlas $\{(B_\kappa, z_\kappa)\}$ for $X(2)$ to an atlas for Y.

Let κ be a cusp of $X(2)$. The branched cover $\pi^{-1}(B_\kappa) \longrightarrow B_\kappa$ restricts to a finite degree topological cover $\pi^{-1}(\dot{B}_\kappa) \longrightarrow \dot{B}_\kappa$. In particular, the composed morphism

$$\pi^{-1}\dot{B}_\kappa \longrightarrow \dot{B}_\kappa \xrightarrow[z_\kappa|_{\dot{B}_\kappa}]{\sim} \dot{B}(0,1)$$

is a finite degree topological cover of $\dot{B}(0,1)$.

Recall that the fundamental group of $\dot{B}(0,1)$ is isomorphic to $\mathbf{Z}$. More precisely, for any connected topological cover of $V \to \dot{B}(0,1)$, there is a unique integer $e \geq 1$ such that $V \to \dot{B}(0,1)$ is isomorphic to the cover $\dot{B}(0,1) \longrightarrow \dot{B}(0,1)$ given by $x \mapsto x^e$.

For every cusp y of Y lying over κ, let $\dot{V}_y$ be the unique connected component of $\pi^{-1}\dot{B}_\kappa$ whose closure V_y in $\pi^{-1}(B_\kappa)$ contains y. Then, for any cusp y, there is a positive integer e_y and an isomorphism

$$w_y : \dot{V}_y \xrightarrow{\sim} \dot{B}(0,1)$$

such that $w_y^{e_y} = z_\kappa \circ \pi|_{\dot{V}_y}$. The isomorphism $w_y : \dot{V}_y \longrightarrow \dot{B}(0,1)$ extends to an isomorphism $w_y : V_y \longrightarrow B(0,1)$ such that $w_y^{e_y} = z_\kappa \circ \pi|_{V_y}$. This shows that e_y is the ramification index of y over κ. Note that we have constructed an atlas $\{(V_y, w_y)\}$ for Y, where y runs over the cusps of Y.

2.3.3. The Arakelov $(1,1)$-form and the hyperbolic metric

Let

$$\mu_{\mathrm{hyp}}(\tau) = \frac{i}{2}\frac{1}{\Im(\tau)^2} d\tau d\overline{\tau}$$

be the hyperbolic $(1,1)$-form on $\mathbf{H}$. A Fuchsian group is a discrete subgroup of $\mathrm{SL}_2(\mathbf{R})$. For any Fuchsian group Γ, the quotient space $\Gamma\backslash\mathbf{H}$ is a connected Hausdorff topological space and can be made into a Riemann surface in a natural way. The hyperbolic metric μ_{hyp} on $\mathbf{H}$ induces a measure on $\Gamma\backslash\mathbf{H}$, given by a smooth positive real-valued $(1,1)$-form outside the set of fixed points of elliptic elements of Γ. If the volume of $\Gamma\backslash\mathbf{H}$ with respect to this measure is finite, we call Γ a *cofinite Fuchsian group*.

Let Γ be a cofinite Fuchsian group, and let X be the compactification of $\Gamma\backslash\mathbf{H}$ obtained by adding the cusps. We assume that Γ has no elliptic elements and that the genus g of X is positive. There is a unique smooth function $F_\Gamma : X \longrightarrow [0,\infty)$ which vanishes at the cusps of Γ such that

$$\mu = \frac{1}{g} F_\Gamma \mu_{\mathrm{hyp}}. \tag{2.3.1}$$

A detailed description of F_Γ is not necessary for our purposes.

Definition 2.3.5. Let $\pi : Y \longrightarrow X(2)$ be a Belyi cover. Then we define the cofinite Fuchsian group Γ_Y (or simply Γ) associated to $\pi : Y \to X(2)$ as follows. Since the topological fundamental group of $Y(2)$ equals

$$\Gamma(2)/\{\pm 1\},$$

we have $\pi^{-1}(Y(2)) = \Gamma'\backslash\mathbf{H}$ for some subgroup $\Gamma' \subset \Gamma(2)/\{\pm1\}$ of finite index. We define $\Gamma \subset \Gamma(2)$ to be the inverse image of Γ' under the quotient map $\Gamma(2) \longrightarrow \Gamma(2)/\{\pm 1\}$. Note that Γ is a cofinite Fuchsian group without elliptic elements.

Theorem 2.3.6. (Jorgenson-Kramer) *For any Belyi cover $\pi : Y \longrightarrow X(2)$, where Y has positive genus,*

$$\sup_{\tau\in Y} F_\Gamma \leq 64 \max_{y\in Y}(e_y)^2 \leq 64(\deg\pi)^2.$$

Proof. This is shown by Bruin in [8]. More precisely, in the notation of *loc. cit.*, Bruin shows that, with $a = 1.44$, we have $N_{\mathrm{SL}_2(\mathbf{Z})}(z, 2a^2-1) \leq 58$. In particular, $\sup_{z\in Y} N_\Gamma(z,z,2a^2-1) \leq 58$; see Section 8.2 in *loc. cit.*. Now, we apply Proposition 6.1 and Lemma 6.2 (with $\epsilon = 2\deg\pi$) in *loc. cit.* to deduce the sought inequality. □

Remark 2.3.7. Jorgenson and Kramer prove a stronger (*albeit* non-explicit) version of Theorem 2.3.6; see [32].

2.3.4. A Merkl atlas for a Belyi cover of $X(2)$

In this section we prove bounds for Arakelov-Green functions of Belyi covers.

Recall that we constructed an atlas $\{(B_\kappa, z_\kappa)\}_\kappa$ for $X(2)$. For a cusp κ of $X(2)$, let

$$y_\kappa : \mathbf{H} \longrightarrow (0, \infty)$$

be defined by

$$\tau \mapsto \Im(\gamma_\kappa^{-1}\tau) = \frac{1}{2} - \frac{\log|z_\kappa(\tau)|}{\pi}.$$

This induces a function $\dot{B}_\kappa \longrightarrow (0, \infty)$ also denoted by y_κ.

Lemma 2.3.8. *For any two cusps κ and κ' of $X(2)$, we have*

$$\left|\frac{dz_\kappa}{dz_{\kappa'}}\right| \leq 4\exp(3\pi/2)$$

on $B_\kappa \cap B_{\kappa'}$.

Proof. We work on the complex upper half-plane $\mathbf{H}$. We may and do assume that $\kappa \neq \kappa'$. By applying $\gamma_{\kappa'}^{-1}$, we may and do assume that $\kappa' = \infty$. On $B_\kappa \cap B_\infty$, we have

$$dz_\kappa(\tau) = \pi i \exp(\pi i \gamma_\kappa^{-1}\tau + \pi/2) d(\gamma_\kappa^{-1}\tau),$$

and

$$dz_\infty(\tau) = \pi i \exp(\pi i \tau + \pi/2) d(\tau).$$

Therefore,

$$\frac{dz_\kappa}{dz_\infty}(\tau) = \exp(\pi i(\gamma_\kappa^{-1}\tau - \tau))\frac{d(\gamma_\kappa^{-1}\tau)}{d(\tau)}.$$

It follows from a simple calculation that, for $\gamma_\kappa^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0$,

$$\left|\frac{dz_\kappa}{dz_\infty}\right|(\tau) = \frac{1}{|c\tau + d|^2}\exp(\pi(y_\infty(\tau) - y_\kappa(\tau))).$$

For τ and $\gamma_\kappa^{-1}\tau$ in B_∞, one has $y_\infty(\tau) > 1/2$ and $y_\kappa(\tau) > 1/2$. From the inequality $|c\tau + d| \geq y_\infty(\tau) = \Im(\tau)$, it follows that

$$y_\kappa(\tau) = \Im(\gamma_\kappa^{-1}(\tau)) = \gamma_\infty\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{\Im\tau}{|c\tau + d|^2} \leq \frac{\Im\tau}{(\Im\tau)^2} \leq 2,$$

and similarly $y_\infty(\tau) \leq 2$. The statement follows. □

Let $\pi : Y \longrightarrow X(2)$ be a Belyi cover. Recall that we constructed an atlas $\{(V_y, w_y)\}$ for Y. We assume that the genus g of Y is positive and, as usual, we let μ denote the Arakelov $(1,1)$-form on Y. Also, we let $V = \pi^{-1}(Y(2))$.

Lemma 2.3.9. *For a cusp y of $\pi : Y \to X(2)$ with $\kappa = \pi(y)$, the equality*

$$idw_y d\overline{w_y} = \frac{2\pi^2 y_\kappa^2 |w_y|^2}{e_y^2} \mu_{\mathrm{hyp}}$$

holds on $\dot{V}_y$.

Proof. Let $\kappa = \pi(y)$ in $X(2)$. We work on the complex upper half-plane. By the chain rule, we have

$$d(z_\kappa) = d(w_y^{e_y}) = e_y w_y^{e_y - 1} dw_y.$$

Therefore,

$$e_y^2 |w_y|^{2e_y - 2} dw_y d\overline{w_y} = dz_\kappa d\overline{z_\kappa}.$$

Note that $dz_\kappa = \pi i z_\kappa d(\gamma_\kappa^{-1})$, where we view $\gamma_\kappa^{-1} : \mathbf{H} \longrightarrow \mathbf{C}$ as a function. Therefore,

$$e_y^2 |w_y|^{2e_y - 2} dw_y d\overline{w_y} = \pi^2 |z_\kappa|^2 d(\gamma_\kappa^{-1}) d(\overline{\gamma_\kappa^{-1}}).$$

Since $|w_y^{e_y}| = |z_\kappa|$, we have

$$\begin{aligned} idw_y d\overline{w_y} &= \frac{i\pi^2 |w_y|^2}{e_y^2} d(\gamma_\kappa^{-1}) d(\overline{\gamma_\kappa^{-1}}) \\ &= \frac{2\pi^2 y_\kappa^2 |w_y|^2}{e_y^2} \frac{id(\gamma_\kappa^{-1}) d(\overline{\gamma_\kappa^{-1}})}{2y_\kappa^2} = \frac{2\pi^2 y_\kappa^2 |w_y|^2}{e_y^2} \left(\mu_{\mathrm{hyp}} \circ \gamma_\kappa^{-1}\right). \end{aligned}$$

Since the hyperbolic (1,1)-form μ_{hyp} is invariant under the action of $\mathrm{SL}_2(\mathbf{Z})$, this concludes the proof. □

Proposition 2.3.10. *Let y be a cusp of $\pi : Y \to X(2)$. Write*

$$\mu = iF_y dw_y d\overline{w_y}$$

on V_y. Then F_y is a subharmonic function on V_y and

$$0 \leq F_y \leq \frac{128 \exp(3\pi)(\deg \pi)^4}{\pi^2 g}.$$

Proof. The first statement follows from [32, page 8]; see also [10, page 58]. The lower bound for F_y is clear from the definition. Let us prove the upper bound for F_y.

For a cusp κ of $X(2)$, let $\dot{B}_\kappa(2) \subset \dot{B}_\kappa$ be the image of the strip

$$\{x + iy : -1 \leq x < 1, y > 2\}$$

in $Y(2)$ under the map $\mathbf{H} \longrightarrow Y(2)$ given by $\tau \mapsto \Gamma(2)\gamma_\kappa \tau$. For a cusp y of Y lying over κ, define $\dot{V}_y(2) = \pi^{-1}(\dot{B}_\kappa(2))$ and $V_y(2) = \dot{V}_y(2) \cup \{y\}$. Since the boundary $\partial V_y(2)$ of $V_y(2)$ is contained in $V_y - V_y(2)$, by the maximum principle for subharmonic functions,

$$\begin{aligned}\sup_{V_y} F_y &= \max(\sup_{V_y(2)} F_y, \sup_{V_y - V_y(2)} F_y) \\ &= \max(\sup_{\partial V_y(2)} F_y, \sup_{V_y - V_y(2)} F_y) \\ &= \sup_{V_y - V_y(2)} F_y.\end{aligned}$$

By Lemma 2.3.9, Definition 2.3.5 and (2.3.1) in Section 2.3.3,

$$F_y = F_\Gamma \frac{e_y^2}{2g\pi^2 y_\kappa^2 |w_y|^2}. \tag{2.3.2}$$

Note that $y_\kappa^{-2} < 4$ on V_y. Furthermore,

$$\sup_{V_y - V_y(2)} |w_y|^{-2} \leq \sup_{B_\kappa - B_\kappa(2)} |z_\kappa|^{-2} = \exp(-\pi) \sup_{B_\kappa - B_\kappa(2)} \exp(2\pi y_\kappa) \leq e^{3\pi}.$$

Thus, the proposition follows from Jorgenson-Kramer's upper bound for F_Γ (Theorem 2.3.6). □

Definition 2.3.11. Define $s_1 = \sqrt{1/2}$. Note that $\frac{1}{2} < s_1 < 1$. For any cusp κ of $X(2)$, let $B_\kappa^{s_1}$ be the open subset of B_κ whose image under z_κ is $\{x \in \mathbf{C} : |x| < s_1\}$. Moreover, define the positive real number r_1 by the equation $r_1^{\deg \pi} = s_1$. Note that $\frac{1}{2} < r_1 < 1$. For all cusps y of $\pi : Y \to X(2)$, define the subset $V_y^{r_1} \subset V_y$ by $V_y^{r_1} = \{x \in V_y : |w_y(x)| < r_1\}$.

Theorem 2.3.12. *Let $\pi : Y \longrightarrow X(2)$ be a Belyi cover such that Y is of genus $g \geq 1$. Then*

$$\sup_{Y \times Y \setminus \Delta} \mathrm{gr}_Y \leq 6378027 \frac{(\deg \pi)^5}{g}.$$

Moreover, for every cusp y and all $x \neq x'$ in $V_y^{r_1}$,

$$|\mathrm{gr}_Y(x, x') - \log |w_y(x) - w_y(x')|| \leq 6378027 \frac{(\deg \pi)^5}{g}$$

Proof. Write $d = \deg \pi$. Let s_1 and r_1 be as in Definition 2.3.11. We define real numbers

$$n := \#(Y - V), \quad M := 4d \exp(3\pi), \quad c_1 := \frac{128 \exp(3\pi) d^4}{\pi^2 g}.$$

Since n is the number of cusps of Y, we have $n \leq 3d$. Moreover

$$\frac{1}{1 - r_1} \leq \frac{d}{1 - s_1}.$$

Note that

$$\frac{330n}{(1 - r_1)^{3/2}} \log \frac{1}{1 - r_1} + 13.2nc_1 + (n - 1) \log M \leq 6378027 \frac{d^5}{g}$$

Therefore, by Theorem 2.3.2, it suffices to show that

$$(\{(V_y, w_y)\}_y, r_1, M, c_1),$$

where y runs over the cusps of $\pi : Y \to X(2)$, constitutes a Merkl atlas for Y.

The first condition of Merkl's theorem is satisfied. That is, $w_y V_y$ is the open unit disc in $\mathbf{C}$.

To verify the second condition of Merkl's theorem, we have to show that the open sets $V_y^{r_1}$ cover Y. For any $x \in V_y$, we have $x \in V_y^{r_1}$ if $\pi(x) \in B_\kappa^{s_1}$. In fact, for any x in V_y, we have $|w_y(x)| < r_1$ if and only if

$$|z_\kappa(\pi(x))| = |w_y(x)|^{e_y} < r_1^{e_y}.$$

Since $r_1 < 1$, we see that $s_1 = r_1^d \leq r_1^{e_y}$. Therefore, if $\pi(x)$ lies in $B_\kappa^{s_1}$, we see that x lies in $V_y^{r_1}$. Now, since $s_1 < \frac{\sqrt{3}}{2}$, we have $X(2) = \cup_{\kappa \in \{0,1,\infty\}} B_\kappa^{s_1}$. Thus, we conclude that $Y = \cup_y V_y^{r_1}$, where y runs through the cusps.

Since we have already verified the fourth condition of Merkl's theorem in Lemma 2.3.10, it suffices to verify the third condition to finish the proof. Let κ and κ' be cusps of $X(2)$. We may and do assume that $\kappa \neq \kappa'$. Now, as usual, we work on the complex upper half-plane. By the chain rule,

$$\left|\frac{dw_y}{dw_{y'}}\right| \leq \frac{d}{|w_y|^{e_y - 1}} \sup_{B_\kappa \cap B_{\kappa'}} \left|\frac{dz_\kappa}{dz_{\kappa'}}\right|$$

on $V_y \cap V_{y'}$. Note that $|w_y(\tau)|^{e_y-1} \geq |w_y(\tau)|^{e_y} = |z_\kappa(\tau)|$ for any τ in $\mathbf{H}$. Therefore,

$$\left|\frac{dw_y}{dw_{y'}}\right| \leq \frac{d}{|z_\kappa|} \sup_{B_\kappa \cap B_{\kappa'}} \left|\frac{dz_\kappa}{dz_{\kappa'}}\right| \leq M,$$

where we used Lemma 2.3.8 and the inequality $|z_\kappa| > \exp(-3\pi/2)$ on $B_\kappa \cap B_{\kappa'}$. □

2.3.5. The Arakelov norm of the Wronskian differential

Proposition 2.3.13. *Let $\pi : Y \longrightarrow X(2)$ be a Belyi cover with Y of genus $g \geq 1$. Then*

$$\sup_{Y - \mathrm{Supp}\mathcal{W}} \log \|\mathrm{Wr}\|_{\mathrm{Ar}} \leq 6378028 g (\deg \pi)^5.$$

Proof. Let b be a non-Weierstrass point on Y and let y be a cusp of Y such that b lies in $V_y^{r_1}$. Let $\omega = (\omega_1, \ldots, \omega_g)$ be an orthonormal basis of $\mathrm{H}^0(Y, \Omega_Y^1)$. Then, as in Section 1.1,

$$\log \|\mathrm{Wr}\|_{\mathrm{Ar}}(b) = \log |W_{w_y}(\omega)(b)| + \frac{g(g+1)}{2} \log \|dw_y\|_{\mathrm{Ar}}(b).$$

By Theorem 2.3.12,

$$\frac{g(g+1)}{2} \log \|dw_y\|_{\mathrm{Ar}}(b) \quad \leq \quad 6378027 g (\deg \pi)^5.$$

Let us show that $\log |W_{w_y}(\omega)(b)| \leq g(\deg \pi)^5$. Write $\omega_k = f_k dw_y$ on V_y. Note that

$$\omega_k \wedge \overline{\omega_k} = |f_k|^2 dw_y \wedge d\overline{w_y}.$$

Therefore,

$$\mu = \frac{i}{2g} \sum_{k=1}^{g} \omega_k \wedge \overline{\omega_k} = \frac{i}{2g} \sum_{k=1}^{g} |f_k|^2 dw_y \wedge d\overline{w_y}.$$

We deduce that $\sum_{k=1}^{g} |f_k|^2 = 2gF_y$, where F_y is the unique function on V_y such that

$$\mu = iF_y dw_y \wedge d\overline{w_y}.$$

By our upper bound for F_y (Proposition 2.3.10), for any $j = 1, \ldots, g$,

$$\sup_{V_y} |f_j|^2 \leq \sup_{V_y} \sum_{k=1}^{g} |f_k|^2 = 2gF_y \leq \frac{256 \exp(3\pi)(\deg \pi)^4}{\pi^2}.$$

By Hadamard's inequality,

$$\log |W_{w_y}(\omega)(b)| \leq \sum_{l=0}^{g-1} \log \left(\sum_{k=1}^{g} \left| \frac{d^l f_k}{dw_y^l} \right|^2 (b) \right)^{1/2}.$$

Let $r_1 < r < 1$ be some real number. By Cauchy's integral formula, for any $0 \leq l \leq g-1$,

$$\left| \frac{d^l f_k}{dw_y^l} \right| (b) = \left| \frac{l!}{2\pi i} \int_{|w_y|=r} \frac{f_k}{(w_y - w_y(b))^{l+1}} dw_y \right|.$$

It is not hard to see that

$$\left| \frac{l!}{2\pi i} \int_{|w_y|=r} \frac{f_k}{(w_y - w_y(b))^{l+1}} dw_y \right| \leq \frac{l!}{(r - r_1)^{l+1}} \sup_{V_y} |f_k| \leq \frac{g!}{(1 - r_1)^g} \sup_{V_y} |f_k|.$$

By the preceding estimations, since $g! \leq g^g$ and $\frac{1}{1-r_1} \leq \frac{\deg \pi}{1-s_1}$, we obtain that

$$\begin{aligned} \log |W_{w_y}(\omega)(b)| &\leq \sum_{l=0}^{g-1} \log \left(\frac{g!}{(1-r_1)^g} \left(\sum_{k=1}^{g} \sup_{V_y} |f_k|^2 \right)^{1/2} \right) \\ &\leq \sum_{l=0}^{g-1} \log \left(\frac{g!}{(1-r_1)^g} \left(\sum_{k=1}^{g} \frac{256 \exp(3\pi)(\deg \pi)^4}{\pi^2} \right)^{1/2} \right). \end{aligned}$$

Note that the latter expression equals

$$g \log(g!) + g^2 \log \left(\frac{1}{1 - r_1} \right) + \frac{g}{2} \log \left(\frac{256 g \exp(3\pi)}{\pi^2} \right) + 2g \log(\deg \pi).$$

Now, note that the latter expression is at most

$$\left(4.5 + \log \left(\frac{1}{1 - s_1} \right) + \frac{1}{2} \log \left(\frac{256 \exp(3\pi)}{\pi^2} \right) \right) g^2 \log(\deg \pi).$$

The latter expression is easily seen to be bounded by

$$13g(\deg \pi)^2.$$

Since $g \geq 1$ and $\pi : Y \to X(2)$ is a Belyi cover, the inequality $\deg \pi \geq 3$ holds. Thus,

$$13g(\deg \pi)^2 \leq \frac{13g(\deg \pi)^5}{27} \leq g(\deg \pi)^5. \qquad \square$$

2.4. Arithmetic part

2.4.1. Lenstra's generalization of Dedekind's discriminant bound

Let A be a discrete valuation ring of characteristic zero with fraction field K. Let ord_A denote the valuation on A. Let L/K be a finite field extension of degree n, and let B be the integral closure of A in L. Note that L/K is separable, and B/A is finite.

The inverse different $\mathfrak{D}_{B/A}^{-1}$ of B over A is the fractional ideal

$$\{x \in L : \mathrm{Tr}(xB) \subset A\},$$

where Tr is the trace of L over K. The inverse of the inverse different, denoted by $\mathfrak{D}_{B/A}$, is the different of B over A. Note that $\mathfrak{D}_{B/A}$ is actually an integral ideal of L.

The following proposition (which we would like to attribute to H.W. Lenstra jr.) is a generalization of Dedekind's discriminant bound; see [56, Proposition III.6.13].

Proposition 2.4.1. (H.W. Lenstra jr.) *Suppose that B is a discrete valuation ring of ramification index e over A. Then, the valuation r of the different ideal $\mathfrak{D}_{B/A}$ on B satisfies the inequality*

$$r \leq e - 1 + e \cdot \mathrm{ord}_A(n).$$

Proof. Let x be a uniformizer of A. Since A is of characteristic zero, we may define $y := \frac{1}{nx}$; note that y is an element of K. The trace of y (as an element of L) is $\frac{1}{x}$. Since $1/x$ is not in A, this implies that the inverse different $\mathfrak{D}_{B/A}^{-1}$ is strictly contained in the fractional ideal yB. (If not, since A and B are discrete valuation rings, we would have that yB is strictly contained in the inverse different.) In particular, the different $\mathfrak{D}_{B/A}$ strictly contains the fractional ideal (nx). Therefore, the valuation $\mathrm{ord}_B(\mathfrak{D}_{B/A})$ on B of $\mathfrak{D}_{B/A}$ is strictly less than the valuation of nx. Thus,

$$\mathrm{ord}_B(\mathfrak{D}_{B/A}) < \mathrm{ord}_B(nx) = e \cdot \mathrm{ord}_A(nx) = e(\mathrm{ord}_A(n) + 1) = e \cdot \mathrm{ord}_A(n) + e.$$

This concludes the proof of the inequality. $\square$

Remark 2.4.2. If the extension of residue fields of B/A is separable, the above lemma follows from the *Remarque* following Proposition III.6.13 in [56]. (The result in *loc. cit.* was conjectured by Dedekind and proved by Hensel when $A = \mathbf{Z}$.) The reader will see that, in the proof of Proposition 2.4.7, we have to deal with imperfect residue fields.

Proposition 2.4.3. *Suppose that the residue characteristic p of A is positive. Let m be the biggest integer such that $p^m \leq n$. Then, for $\beta \subset B$ a maximal ideal of B with ramification index e_β over A, the valuation r_β of the different ideal $\mathfrak{D}_{B/A}$ at β satisfies the inequality*

$$r_\beta \leq e_\beta - 1 + e_\beta \cdot \mathrm{ord}_A(p^m).$$

Proof. To compute r_β, we localize B at β, and then take the completions $\widehat{A}$ and $\widehat{B_\beta}$ of A and B_β, respectively. Let d be the degree of $\widehat{B_\beta}$ over $\widehat{A}$. Then, by Lenstra's result (Proposition 2.4.1), the inequality

$$r_\beta \leq e_\beta - 1 + e_\beta \cdot \mathrm{ord}_{\widehat{A}}(d).$$

holds. By definition, $\mathrm{ord}_{\widehat{A}}(d) = \mathrm{ord}_A(d) \leq \mathrm{ord}_A(p^m)$. This concludes the proof. □

2.4.2. Covers of arithmetic surfaces with fixed branch locus

Let K be a number field with ring of integers O_K, and let $S = \mathrm{Spec}\, O_K$. Let D be a reduced effective divisor on $\mathcal{X} = \mathbf{P}^1_S$, and let U denote the complement of the support of D in $\mathcal{X}$. Let $\mathcal{Y} \to S$ be an integral normal 2-dimensional flat projective S-scheme with geometrically connected fibres, and let $\pi : \mathcal{Y} \longrightarrow \mathcal{X}$ be a finite surjective morphism of S-schemes which is étale over U. Let $\psi : \mathcal{Y}' \to \mathcal{Y}$ be the minimal resolution of singularities ([41, Proposition 9.3.32]). Note that we have the following diagram of morphisms

$$\mathcal{Y}' \xrightarrow{\psi} \mathcal{Y} \xrightarrow{\pi} \mathcal{X} \longrightarrow S.$$

Consider the prime decomposition $D = \sum_{i \in I} D_i$, where I is a finite index set. Let D_{ij} be an irreducible component of $\pi^{-1}(D)$ mapping onto D_i, where j is in the index set J_i. We define r_{ij} to be the valuation of the different ideal of $\mathcal{O}_{\mathcal{Y},D_{ij}}/\mathcal{O}_{\mathcal{X},D_i}$. We define the ramification divisor R to be $\sum_{i\in I}\sum_{j\in J_i} r_{ij}D_{ij}$. We define $B := \pi_* R$.

We apply [41, 6.4.26] to obtain that there exists a dualizing sheaf $\omega_{\mathcal{Y}/S}$ for $\mathcal{Y} \to S$, and a dualizing sheaf ω_π for $\pi : \mathcal{Y} \to \mathcal{X}$ such that the adjunction formula

$$\omega_{\mathcal{Y}/S} = \pi^* \omega_{\mathcal{X}/S} \otimes \omega_\pi$$

holds. Since the local ring at the generic point of a divisor on $\mathcal{X}$ is of characteristic zero, basic properties of the different ideal imply that ω_π is canonically isomorphic to the line bundle $\mathcal{O}_{\mathcal{Y}}(R)$.

We deduce the *Riemann-Hurwitz* formula

$$\omega_{\mathcal{Y}/S} = \pi^* \omega_{\mathcal{X}/S} \otimes \mathcal{O}_{\mathcal{Y}}(R).$$

Let $K_{\mathcal{X}} = -2 \cdot [\infty]$ be the divisor defined by the tautological section of $\omega_{\mathcal{X}/O_K}$. Let $K_{\mathcal{Y}'}$ denote the Cartier divisor on $\mathcal{Y}'$ defined by the rational section $d(\pi \circ \psi)$ of $\omega_{\mathcal{Y}'/S}$. We define the Cartier divisor $K_{\mathcal{Y}}$ on $\mathcal{Y}$ analogously, i.e., $K_{\mathcal{Y}}$ is the Cartier divisor on $\mathcal{Y}$ defined by $d\pi$. Note that $K_{\mathcal{Y}} = \psi_* K_{\mathcal{Y}'}$. Also, the Riemann-Hurwitz formula implies the following equality of Cartier divisors

$$K_{\mathcal{Y}} = \pi^* K_{\mathcal{X}} + R.$$

Let $E_1, \ldots, E_s$ be the exceptional components of $\psi : \mathcal{Y}' \longrightarrow \mathcal{Y}$. Note that the pull-back of the Cartier divisor $\psi^* K_{\mathcal{Y}}$ coincides with $K_{\mathcal{Y}'}$ on

$$\mathcal{Y}' - \bigcup_{i=1}^{s} E_i.$$

Therefore, there exist integers c_i such that

$$K_{\mathcal{Y}'} = \psi^* K_{\mathcal{Y}} + \sum_{i=1}^{s} c_i E_i,$$

where this is an equality of Cartier divisors (**not only** modulo linear equivalence). Note that $(\psi^* K_{\mathcal{Y}}, E_i) = 0$ for all i. In fact, $K_{\mathcal{Y}}$ is linearly equivalent to a Cartier divisor with support disjoint from the singular locus of $\mathcal{Y}$.

Lemma 2.4.4. *For all* $i = 1, \ldots, s$*, we have* $c_i \leq 0$.

Proof. We have the following local statement. Let y be a singular point of $\mathcal{Y}$, and let $E_1, \ldots, E_r$ be the exceptional components of ψ lying over y. We define

$$V_+ = \sum_{i=1, c_i > 0}^{r} c_i E_i$$

as the sum on the $c_i > 0$. To prove the lemma, it suffices to show that $V_+ = 0$. Since the intersection form on the exceptional locus of $\mathcal{Y}' \to \mathcal{Y}$ is negative definite ([41, Proposition 9.1.27]), to prove $V_+ = 0$, it suffices to show that $(V_+, V_+) \geq 0$. Clearly, to prove the latter inequality, it suffices to show that, for all i such that $c_i > 0$, we have $(V_+, E_i) \geq 0$. To do this, fix $i \in \{1, \ldots, r\}$ with $c_i > 0$. Since $\mathcal{Y}' \to \mathcal{Y}$ is minimal, we have that E_i is not a (-1)-curve. In particular, by the adjunction formula, the inequality $(K_{\mathcal{Y}'}, E_i) \geq 0$ holds. We conclude that

$$(V_+, E_i) = (K_{\mathcal{Y}'}, E_i) - \sum_{j=1, c_j < 0}^{r} c_j (E_j, E_i) \geq 0,$$

where, in the last inequality, we used that, for all j such that $c_j < 0$, we have that $E_j \neq E_i$. $\square$

Proposition 2.4.5. *Let $P' : S \to \mathcal{Y}'$ be a section, and let $Q : S \to \mathcal{X}$ be the induced section. If the image of P' is not contained in the support of $K_{\mathcal{Y}'}$, then*

$$(K_{\mathcal{Y}'}, P')_{\text{fin}} \leq (B, Q)_{\text{fin}}.$$

Proof. Note that, by the Riemann-Hurwitz formula, we have $K_{\mathcal{Y}} = \pi^* K_{\mathcal{X}} + R$. Therefore, by Lemma 2.4.4, we get that

$$\begin{aligned} (K_{\mathcal{Y}'}, P')_{\text{fin}} &= (\psi^* K_{\mathcal{Y}} + \sum c_i E_i, P')_{\text{fin}} \\ &= (\psi^* \pi^* K_{\mathcal{X}} + \psi^* R + \sum_{i=1}^{s} c_i E_i, P')_{\text{fin}} \\ &\leq (\psi^* \pi^* K_{\mathcal{X}}, P')_{\text{fin}} + (\psi^* R, P')_{\text{fin}}. \end{aligned}$$

Since the image of P' is not contained in the support of $K_{\mathcal{Y}'}$, we can apply the projection formula for the composed morphism $\pi \circ \psi : \mathcal{Y}' \to \mathcal{X}$ to $(\psi^* \pi^* K_{\mathcal{X}}, P')_{\text{fin}}$ and $(\psi^* R, P')_{\text{fin}}$; see [41, Section 9.2]. This gives

$$(K_{\mathcal{Y}'}, P')_{\text{fin}} \leq (\psi^* \pi^* K_{\mathcal{X}}, P')_{\text{fin}} + (\psi^* R, P')_{\text{fin}} = (K_{\mathcal{X}}, Q)_{\text{fin}} + (\pi_* R, Q)_{\text{fin}}.$$

Since $K_{\mathcal{X}} = -2 \cdot [\infty]$, the inequality $(K_{\mathcal{X}}, Q)_{\text{fin}} \leq 0$ holds. By definition, $B = \pi_* R$. This concludes the proof. □

We introduce some notation. For i in I and j in J_i, let e_{ij} and f_{ij} be the ramification index and residue degree of π at the generic point of D_{ij}, respectively. Moreover, let $\mathfrak{p}_i \subset O_K$ be the maximal ideal corresponding to the image of D_i in $\operatorname{Spec} O_K$. Then, note that e_{ij} is the multiplicity of D_{ij} in the fibre of $\mathcal{Y}$ over $\mathfrak{p}_i$. Now, let $e_{\mathfrak{p}_i}$ and $f_{\mathfrak{p}_i}$ be the ramification index and residue degree of $\mathfrak{p}_i$ over $\mathbf{Z}$, respectively. Finally, let p_i be the residue characteristic of the local ring at the generic point of D_i and, if $p_i > 0$, let m_i be the biggest integer such that $p_i^{m_i} \leq \deg \pi$, i.e., $m_i = \lfloor \log(\deg \pi)/\log(p_i) \rfloor$.

Lemma 2.4.6. *Let i be in I such that $0 < p_i \leq \deg \pi$. Then, for all j in J_i,*

$$r_{ij} \leq 2 e_{ij} m_i e_{\mathfrak{p}_i}.$$

Proof. Let ord_{D_i} be the valuation on the local ring at the generic point of D_i. Then, by Lenstra's result (Proposition 2.4.3), the inequality

$$r_{ij} \leq e_{ij} - 1 + e_{ij} \cdot \operatorname{ord}_{D_i}(p_i^{m_i})$$

holds. Note that $\operatorname{ord}_{D_i}(p_i^{m_i}) = m_i e_{\mathfrak{p}_i}$. Since $p_i \leq \deg \pi$, we have that $m_i \geq 1$. Therefore,

$$r_{ij} \leq e_{ij} - 1 + e_{ij} m_i e_{\mathfrak{p}_i} \leq 2 e_{ij} m_i e_{\mathfrak{p}_i}.$$

□

Let us introduce a bit more notation. Let I_1 be the set of i in I such that D_i is horizontal (i.e., $p_i = 0$) or $p_i > \deg \pi$. Let $D_1 = \sum_{i\in I_1} D_i$. We are now finally ready to combine our results to bound the "non-archimedean" part of the height of a point.

Proposition 2.4.7. *Let $P' : S \to \mathcal{Y}'$ be a section, and let $Q : S \to \mathcal{X}$ be the induced section. If the image of P' is not contained in the support of $K_{\mathcal{Y}'}$, then*

$$(K_{\mathcal{Y}'}, P')_{\mathrm{fin}} \leq \deg \pi (D_1, Q)_{\mathrm{fin}} + 2(\deg \pi)^2 \log(\deg \pi)[K : \mathbf{Q}].$$

Proof. Note that

$$B = \sum_{i\in I} \left(\sum_{j\in J_i} r_{ij} f_{ij} \right) D_i.$$

Let I_2 be the complement of I_1 in I. Let $D_2 = \sum_{i\in I_2} D_i$, and note that $D = D_1 + D_2$. In particular,

$$\begin{aligned}(B, Q)_{\mathrm{fin}} &= \sum_{i\in I}\sum_{j\in J_i} r_{ij} f_{ij} (D_i, Q)_{\mathrm{fin}} \\ &= \sum_{i\in I_1}\sum_{j\in J_i} r_{ij} f_{ij} (D_i, Q)_{\mathrm{fin}} + \sum_{i\in I_2}\sum_{j\in J_i} r_{ij} f_{ij} (D_i, Q)_{\mathrm{fin}}.\end{aligned}$$

Note that, for all i in I_1 and j in J_i, the ramification of D_{ij} over D_i is tame, i.e., the equality $r_{ij} = e_{ij} - 1$ holds. Note that, for all i in I, we have $\sum_{j\in J_i} e_{ij} f_{ij} = \deg \pi$. Thus,

$$\sum_{i\in I_1}\sum_{j\in J_i} r_{ij} f_{ij} (D_i, Q)_{\mathrm{fin}} \leq \sum_{i\in I_1}\sum_{j\in J_i} e_{ij} f_{ij} (D_i, Q)_{\mathrm{fin}} = \deg \pi (D_1, Q)_{\mathrm{fin}}.$$

We claim that

$$\sum_{i\in I_2}\sum_{j\in J_i} r_{ij} f (D_i, Q)_{\mathrm{fin}} \leq 2(\deg \pi)^2 \log(\deg \pi)[K : \mathbf{Q}].$$

In fact, since, for all i in I_2 and j in J_i, by Proposition 2.4.6, the inequality

$$r_{ij} \leq 2 e_{ij} m_i e_{\mathfrak{p}_i}$$

holds, we have that

$$\begin{aligned}\sum_{i\in I_2}\sum_{j\in J_i} r_{ij} f_{ij} (D_i, Q)_{\mathrm{fin}} &\leq 2 \sum_{i\in I_2} m_i e_{\mathfrak{p}_i} (D_i, Q)_{\mathrm{fin}} \left(\sum_{j\in J_i} e_{ij} f_{ij} \right) \\ &= 2(\deg \pi) \sum_{i\in I_2} m_i e_{\mathfrak{p}_i} (D_i, Q)_{\mathrm{fin}}.\end{aligned}$$

Note that $(D_i, Q) = \log(\# k(\mathfrak{p}_i)) = f_{\mathfrak{p}_i} \log p_i$. We conclude that

$$\begin{aligned}\sum_{i\in I_2} m_i e_{\mathfrak{p}_i} (D_i, Q)_{\mathrm{fin}} &= \sum_{p \text{ prime}} \left(\sum_{i\in I_2, p_i = p} e_{\mathfrak{p}_i} f_{\mathfrak{p}_i} \right) \left\lfloor \frac{\log(\deg \pi)}{\log p} \right\rfloor \log(p) \\ &= [K : \mathbf{Q}] \sum_{\mathcal{X}_p \cap |D_2| \neq \emptyset} \left\lfloor \frac{\log(\deg \pi)}{\log p} \right\rfloor \log(p),\end{aligned}$$

where the last sum runs over all prime numbers p such that the fibre $\mathcal{X}_p$ contains an irreducible component of the support of D_2. Thus, the real number $(B, Q)_{\mathrm{fin}}$ is at most

$$(\deg \pi)(D_1, Q)_{\mathrm{fin}} + 2(\deg \pi)[K : \mathbf{Q}] \sum_{\mathcal{X}_p \cap D_2 \neq \emptyset} \left\lfloor \frac{\log(\deg \pi)}{\log p} \right\rfloor \log(p)$$

holds. Note that

$$\sum_{\mathcal{X}_p \cap D_2 \neq \emptyset} \left\lfloor \frac{\log(\deg \pi)}{\log p} \right\rfloor \log(p) \leq \sum_{\mathcal{X}_p \cap D_2 \neq \emptyset} \log(\deg \pi) \leq \deg \pi \log(\deg \pi),$$

where we used that $\mathcal{X}_p \cap D_2 \neq \emptyset$ implies that $p \leq \deg \pi$. In particular,

$$(B, Q)_{\mathrm{fin}} \leq (\deg \pi)(D_1, Q)_{\mathrm{fin}} + 2(\deg \pi)^2 \log(\deg \pi)[K : \mathbf{Q}].$$

By Proposition 2.4.5, we conclude that

$$(K_{\mathcal{Y}'}, P')_{\mathrm{fin}} \leq (\deg \pi)(D_1, Q)_{\mathrm{fin}} + 2(\deg \pi)^2 \log(\deg \pi)[K : \mathbf{Q}].$$ □

2.4.3. Models of covers of curves

In this section, we give a general construction for a model of a cover of the projective line. Let K be a number field with ring of integers O_K, and let $S = \operatorname{Spec} O_K$.

Proposition 2.4.8. *Let $\mathcal{Y} \to \operatorname{Spec} O_K$ be a flat projective morphism with geometrically connected fibres of dimension one, where $\mathcal{Y}$ is an integral normal scheme. Then, there exists a finite field extension L/K such that the minimal resolution of singularities of the normalization of $\mathcal{Y} \times_{O_K} O_L$ is semi-stable over O_L.*

Proof. This follows from [42, Corollary 2.8]. □

The main result of this section reads as follows.

Theorem 2.4.9. *Let K be a number field, and let Y be a smooth projective geometrically connected curve over K. Then, for any finite morphism $\pi_K : Y \to \mathbf{P}^1_K$, there exists a number field L/K such that:*

- *the normalization $\pi : \mathcal{Y} \to \mathbf{P}^1_{O_L}$ of $\mathbf{P}^1_{O_L}$ in the function field of Y_L is finite flat surjective;*
- *the minimal resolution of singularities $\psi : \mathcal{Y}' \longrightarrow \mathcal{Y}$ is semi-stable over O_L;*
- *each irreducible component of the vertical part of the branch locus of the finite flat morphism $\pi : \mathcal{Y} \to \mathbf{P}^1_{O_L}$ is of characteristic less or equal to $\deg \pi$. (The characteristic of a prime divisor D on $\mathbf{P}^1_{O_L}$ is the residue characteristic of the local ring at the generic point of D.)*

Proof. By Proposition 2.4.8, there exists a finite field extension L/K such that the minimal resolution of singularities $\psi : \mathcal{Y}' \longrightarrow \mathcal{Y}$ of the normalization of $\mathbf{P}^1_{O_L}$ in the function field of Y_L is semi-stable over O_L. Note that the finite morphism $\pi : \mathcal{Y} \to \mathbf{P}^1_{O_L}$ is flat. (The source is normal of dimension two, and the target is regular.) Moreover, since the fibres of $\mathcal{Y}' \to \operatorname{Spec} O_L$ are reduced, the fibres of $\mathcal{Y}$ over O_L are reduced. Let $\mathfrak{p} \subset O_L$ be a maximal ideal of residue characteristic strictly bigger than $\deg \pi$, and note that the ramification of $\pi : \mathcal{Y} \to \mathbf{P}^1_{O_L}$ over (each prime divisor of $\mathbf{P}^1_{O_L}$ lying over) $\mathfrak{p}$ is tame. Since the fibres of $\mathcal{Y} \to \operatorname{Spec} O_L$ are reduced, we see that the finite morphism π is unramified over $\mathfrak{p}$. In fact, since $\mathbf{P}^1_{O_L} \to \operatorname{Spec} O_L$ has reduced (even smooth) fibres, the valuation of the different ideal $\mathcal{D}_{\mathcal{O}_D/\mathcal{O}_{\pi(D)}}$ on $\mathcal{O}_D$ of an irreducible component D of $\mathcal{Y}_{\mathfrak{p}}$ lying over $\pi(D)$ in $\mathcal{X}$ is precisely the multiplicity of D in $\mathcal{Y}_{\mathfrak{p}}$. (Here we let $\mathcal{O}_D$ denote the local ring at the generic point of D, and $\mathcal{O}_{\pi(D)}$ the local ring at the generic point of $\pi(D)$.) Thus, each irreducible component of the vertical part of the branch locus of $\pi : \mathcal{Y} \to \mathbf{P}^1_{O_L}$ is of characteristic less or equal to $\deg \pi$. □

2.5. Proof of main result

2.5.1. The modular lambda function

The modular function $\lambda : \mathbf{H} \to \mathbf{C}$ is defined as

$$\lambda(\tau) = \frac{\wp\left(\frac{1}{2} + \frac{\tau}{2}\right) - \wp\left(\frac{\tau}{2}\right)}{\wp\left(\frac{\tau}{2}\right) - \wp\left(\frac{1}{2}\right)},$$

where $\wp$ denotes the Weierstrass elliptic function for the lattice $\mathbf{Z} + \tau\mathbf{Z}$ in $\mathbf{C}$. The function λ is $\Gamma(2)$-invariant. More precisely, λ factors through the $\Gamma(2)$-quotient map $\mathbf{H} \to Y(2)$ and an analytic isomorphism

$$Y(2) \xrightarrow{\simeq} \mathbf{C} - \{0, 1\}.$$

Thus, the modular function λ induces an analytic isomorphism from $X(2)$ to $\mathbf{P}^1(\mathbf{C})$. Let us note that $\lambda(i\infty) = 0$, $\lambda(1) = \infty$ and $\lambda(0) = 1$.

The restriction of λ to the imaginary axis $\{iy : y > 0\}$ in $\mathbf{H}$ induces a homeomorphism, also denoted by λ, from $\{iy : y > 0\}$ to the open interval $(0, 1)$ in $\mathbf{R}$. In fact, for α in the open interval $(0, 1)$,

$$\lambda^{-1}(\alpha) = i\frac{\mathrm{M}(1, \sqrt{\alpha})}{\mathrm{M}(1, \sqrt{1-\alpha})},$$

where M denotes the arithmetic-geometric-mean.

Lemma 2.5.1. *For τ in $\mathbf{H}$, let $q(\tau) = \exp(\pi i \tau)$ and let*

$$\lambda(\tau) = \sum_{n=1}^{\infty} a_n q^n(\tau)$$

be the q-expansion of λ on $\mathbf{H}$. *Then, for any real number* $4/5 \leq y \leq 1$,

$$-\log |\sum_{n=1}^{\infty} n a_n q^n(iy)| \leq 2.$$

Proof. Note that

$$\sum_{n=1}^{\infty} n a_n q^n = q\frac{d\lambda}{dq}.$$

It suffices to show that $|qd\lambda/dq| \geq 3/20$. We will use the product formula for λ. Namely,

$$\lambda(q) = 16q\prod_{n=1}^{\infty} f_n(q), \quad f_n(q) := \frac{1+q^{2n}}{1+q^{2n-1}}.$$

Write $f_n'(q) = df_n(q)/dq$. Then,

$$q\frac{d\lambda}{dq} = \lambda\left(1 + q\sum_{n=1}^{\infty}\frac{f_n'(q)}{f_n(q)}\right) = \lambda\left(1 + q\sum_{n=1}^{\infty}\frac{d}{dq}(\log f_n(q))\right).$$

Note that, for any positive integer n and $4/5 \leq y \leq 1$,

$$\left(\frac{d}{dq}\log f_n(q)\right)(iy) \leq 0.$$

Moreover, since $\lambda(i) = 1/2$ and $\lambda(0) = 1$, the inequality $\lambda(iy) \geq 1/2$ holds for all $0 \leq y \leq 1$. Also, for all $4/5 \leq y \leq 1$,

$$\left(-q\sum_{n=1}^{\infty}\left(\frac{d}{dq}(\log f_n(q))\right)\right)(iy) \leq \frac{7}{10}.$$

In fact,

$$\sum_{n=1}^{\infty}\frac{d}{dq}(\log f_n(q)) \quad = \quad \sum_{n=1}^{\infty}\frac{2nq^{2n-1}}{1+q^{2n}} - \sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-2}}{1+q^{2n-1}}$$

It is straightforward to verify that, for all $4/5 \leq y \leq 1$, the real number

$$\sum_{n=1}^{\infty}\frac{2nq^{2n-1}(iy)}{1+q^{2n}(iy)} - \sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-2}(iy)}{1+q^{2n-1}(iy)}$$

is at least

$$\frac{100}{109}\sum_{n=1}^{\infty} 2nq^{2n-1}(iy) - \sum_{n=1}^{\infty}(2n-1)q^{2n-2}(iy)$$

holds. Finally, utilizing classical formulas for geometric series, for all $4/5 \leq y \leq 1$, the real number

$$q(iy)\sum_{n=1}^{\infty}\frac{d}{dq}(\log f_n(q))(iy)$$

is at least

$$q(iy)\left(\frac{200q(iy)}{109(1-q^2(iy))^2}-\frac{1+q^2(iy)}{(1-q^2(iy))^2}\right)\geq\frac{7}{10}.$$

We conclude that

$$\left|q\frac{d\lambda}{dq}\right|\geq\frac{1}{2}\left(1-\frac{7}{10}\right)=\frac{3}{20}.$$

□

2.5.2. A non-Weierstrass point with bounded height

The logarithmic height of a non-zero rational number $a=p/q$ is given by

$$h_{\text{naive}}(a)=\log\max(|p|,|q|),$$

where p and q are coprime integers and $q>0$.

Theorem 2.5.2. *Let $\pi_{\overline{\mathbf{Q}}}:Y\longrightarrow\mathbf{P}^1_{\overline{\mathbf{Q}}}$ be a finite morphism of degree d, where $Y/\overline{\mathbf{Q}}$ is a smooth projective connected curve of positive genus $g\geq 1$. Assume that $\pi_{\overline{\mathbf{Q}}}:Y\to\mathbf{P}^1_{\overline{\mathbf{Q}}}$ is unramified over $\mathbf{P}^1_{\overline{\mathbf{Q}}}-\{0,1,\infty\}$. Then, for any rational number $0<a\leq 2/3$ and any $b\in Y(\overline{\mathbf{Q}})$ lying over a,*

$$h(b)\leq 3h_{naive}(a)d^2+6378031\frac{d^5}{g}.$$

Proof. By Theorem 2.4.9, there exist a number field K and a model

$$\pi_K:Y\longrightarrow\mathbf{P}^1_K$$

for $\pi_{\overline{\mathbf{Q}}}:Y\longrightarrow\mathbf{P}^1_{\overline{\mathbf{Q}}}$ with the following three properties: the minimal resolution of singularities $\psi:\mathcal{Y}'\longrightarrow\mathcal{Y}$ of the normalization $\pi:\mathcal{Y}\longrightarrow\mathbf{P}^1_{O_K}$ of $\mathbf{P}^1_{O_K}$ in $\mathcal{Y}$ is semi-stable over O_K, each irreducible component of the vertical part of the branch locus of $\pi:\mathcal{Y}\to\mathbf{P}^1_{O_K}$ is of characteristic less or equal to $\deg\pi$ and every point in the fibre of π_K over a is K-rational. Also, the morphism $\pi:\mathcal{Y}\to\mathbf{P}^1_{O_K}$ is finite flat surjective.

Let $b\in Y(K)$ lie over a. Let P' be the closure of b in $\mathcal{Y}'$. By Lemma 1.7.2, the height of b is "minimal" on the minimal regular model. That is,

$$h(b)\leq\frac{(P',\omega_{\mathcal{Y}'/O_K})}{[K:\mathbf{Q}]}.$$

Recall the following notation from Section 2.4.2. Let $\mathcal{X}=\mathbf{P}^1_{O_K}$. Let $K_{\mathcal{X}}=-2\cdot[\infty]$ be the divisor defined by the tautological section. Let $K_{\mathcal{Y}'}$ be the divisor on $\mathcal{Y}'$ defined by $d(\pi_K)$ viewed as a rational section of $\omega_{\mathcal{Y}'/O_K}$. Since the support of $K_{\mathcal{Y}'}$ on the generic fibre is contained in $\pi_K^{-1}(\{0,1,\infty\})$, the section P' is not contained in the support of $K_{\mathcal{Y}'}$. Therefore, we get that

$$h(b)[K:\mathbf{Q}]\leq(P',\omega_{\mathcal{Y}'/O_K})=(P',K_{\mathcal{Y}'})_{\text{fin}}+\sum_{\sigma:K\longrightarrow\mathbf{C}}(-\log\|d\pi_K\|_\sigma)(\sigma(b)).$$

Let D be the branch locus of $\pi : \mathcal{Y} \longrightarrow \mathcal{X}$ endowed with the reduced closed subscheme structure. Write $D = 0 + 1 + \infty + D_{\text{ver}}$, where D_{ver} is the vertical part of D. Note that, in the notation of Section 2.4.2, we have that $D_1 = 0 + 1 + \infty$. Thus, if Q denotes the closure of a in $\mathcal{X}$, by Proposition 2.4.7, we get

$$(P', K_{\mathcal{Y}'})_{\text{fin}} \leq (\deg \pi)(0 + 1 + \infty, Q)_{\text{fin}} + 2(\deg \pi)^2 \log(\deg \pi)[K : \mathbf{Q}].$$

Write $a = p/q$, where p and q are coprime positive integers with $q > p$. Note that

$$\begin{aligned}(0 + 1 + \infty, Q)_{\text{fin}} &= [K : \mathbf{Q}] \log(pq(q-p)) \\ &\leq 3\log(q)[K : \mathbf{Q}] \\ &= 3h_{\text{naive}}(a)[K : \mathbf{Q}].\end{aligned}$$

We conclude that

$$\frac{(P', K_{\mathcal{Y}'})_{\text{fin}}}{[K : \mathbf{Q}]} \leq 3h_{\text{naive}}(a)(\deg \pi)^2 + 2(\deg \pi)^3.$$

It remains to estimate $\sum_{\sigma: K \longrightarrow \mathbf{C}} (-\log \|d\pi_K\|_\sigma)(\sigma(b))$. To do this, we will use our bounds for Arakelov-Green functions.

Let $\sigma : K \to \mathbf{C}$ be an embedding. The composition

$$Y_\sigma \xrightarrow{\pi_\sigma} \mathbf{P}^1(\mathbf{C}) \xrightarrow{\lambda^{-1}} X(2)$$

is a Belyi cover (Definition 2.3.3). By abuse of notation, let π denote the composed morphism $Y_\sigma \longrightarrow X(2)$. Note that $\lambda^{-1}(2/3) \approx 0.85i$. In particular, $\Im(\lambda^{-1}(a)) \geq \Im(\lambda^{-1}(2/3)) > s_1$. (Recall that $s_1 = \sqrt{1/2}$.) Therefore, the element $\lambda^{-1}(a)$ lies in $\dot{B}_\infty^{s_1}$. Since $V_y^{r_1} \supset V_y \cap \pi^{-1} B_\infty^{s_1}$, there is a unique cusp y of $Y_\sigma \to X(2)$ lying over ∞ such that $\sigma(b)$ lies in $V_y^{r_1}$.

Note that $q = z_\infty \exp(-\pi/2)$. Therefore, since $\lambda = \sum_{j=1}^\infty a_j q^j$ on $\mathbf{H}$,

$$\lambda \circ \pi = \sum_{j=1}^\infty a_j \exp(-j\pi/2)(z_\infty \circ \pi)^j = \sum_{j=1}^\infty a_j \exp(-j\pi/2) w_y^{e_y j}$$

on V_y. Thus, by the chain rule,

$$d(\lambda \circ \pi) = e_y \sum_{j=1}^\infty j a_j \exp(-j\pi/2) w_y^{e_y j - 1} d(w_y).$$

The real number

$$-\log \|d(\lambda \circ \pi)\|_{\text{Ar}}(\sigma(b))$$

equals

$$-\log \|dw_y\|_{\text{Ar}}(\sigma(b)) - \log |e_y \sum_{j=1}^\infty j a_j e^{-j\pi/2} w_y^{e_y j - 1}(\sigma(b))|.$$

By the trivial inequality $e_y \geq 1$ and the inequality $|w_y| \leq 1$, the latter is at most

$$-\log \|dw_y\|_{\mathrm{Ar}}(\sigma(b)) - \log |\sum_{j=1}^{\infty} j a_j e^{-j\pi/2} w_y^{e_y j}(\sigma(b))|.$$

By Lemma 2.5.1, the latter is at most

$$-\log \|dw_y\|_{\mathrm{Ar}}(\sigma(b)) + 2.$$

Thus, by Theorem 2.3.12, we conclude that

$$\frac{\sum_{\sigma:K\to\mathbf{C}}(-\log \|d\pi_K\|_{\sigma})(\sigma(b))}{[K:\mathbf{Q}]} \leq 6378027\frac{(\deg \pi)^5}{g} + 2. \qquad \square$$

Corollary 2.5.3. *Let X be a smooth projective connected curve over $\overline{\mathbf{Q}}$ of genus $g \geq 1$. Then, there exists a point b in $X(\overline{\mathbf{Q}})$ such that*

$$h(b) \leq 6378032\frac{\deg_B(X)^5}{g}.$$

Proof. Write $d = \deg_B(X)$. We apply Theorem 2.5.2 with $a = 1/2$. This gives

$$h(b) \leq 3\log(2)d^2 + 6378031\frac{d^5}{g}.$$

Since $d \geq 3$ and $d \geq g$, the inequality $3\log(2)\deg_B(X)^2 \leq d^5/g$ holds. This clearly implies the sought inequality. $\square$

Theorem 2.5.4. *Let Y be a smooth projective connected curve over $\overline{\mathbf{Q}}$ of genus $g \geq 1$. For any finite morphism $\pi : Y \to \mathbf{P}^1_{\overline{\mathbf{Q}}}$ ramified over exactly three points, there exists a non-Weierstrass point b on Y such that*

$$h(b) \leq 6378033\frac{(\deg \pi)^5}{g}.$$

Proof. Define the sequence $(a_n)_{n=1}^{\infty}$ of rational numbers by $a_1 = 1/2$ and, for all $n \geq 2$, by $a_n = n/(2n-1)$. Note that $1/2 \leq a_n \leq 2/3$, and that the inequality $h_{\mathrm{naive}}(a_n) \leq \log(2n)$ holds. We may and do assume that $\pi : Y \to \mathbf{P}^1_{\overline{\mathbf{Q}}}$ is unramified over $\mathbf{P}^1_{\overline{\mathbf{Q}}} - \{0, 1, \infty\}$. By Theorem 2.5.2, for all $x \in \pi^{-1}(\{a_n\})$,

$$h(x) \leq 3\log(2n)(\deg \pi)^2 + 6378031\frac{(\deg \pi)^5}{g}. \qquad (2.5.1)$$

Since the number of Weierstrass points on Y is at most $g^3 - g$, there exists an integer $1 \leq i \leq (\deg \pi)^2$ such that the fibre $\pi^{-1}(a_i)$ contains a non-Weierstrass point, say b. Applying (2.5.1) to b, we obtain that

$$h(b) \leq 3\log(2(\deg \pi)^2)(\deg \pi)^2 + 6378031\frac{(\deg \pi)^5}{g}.$$

Therefore, we may conclude that

$$h(b) \leq 2\frac{(\deg \pi)^5}{g} + 6378031\frac{(\deg \pi)^5}{g}. \qquad \square$$

Remark 2.5.5. Let us mention that combining the above results with a theorem of Zhang (Theorem4.5.2) we obtain infinitely many points b in $X(\overline{\mathbf{Q}})$ such that

$$h(b) \leq 13 \cdot 10^6 \deg_B(X)^5.$$

2.5.3. Proving Theorem 2.1.1

For a smooth projective connected curve X over $\overline{\mathbf{Q}}$, we let $\deg_B(X)$ denote the Belyi degree of X.

Proof of Theorem 2.1.1. The lower bound for $\Delta(X) \geq 0$ is trivial, the lower bound $e(X) \geq 0$ is due to Faltings ([24, Theorem 5]) and the lower bound $h_{\mathrm{Fal}}(X) \geq -g \log(2\pi)$ is due to Bost (Lemma 2.2.4).

For the remaining bounds, we proceed as follows. By Theorem 2.5.4, there exists a non-Weierstrass point b in $X(\overline{\mathbf{Q}})$ such that

$$h(b) \leq 6378033\frac{\deg_B(X)^5}{g}.$$

By our bound on the Arakelov norm of the Wronskian differential in Proposition 2.3.13, we have

$$\log \|\mathrm{Wr}\|_{\mathrm{Ar}}(b) \leq 6378028g \deg_B(X)^5.$$

To obtain the theorem, we combine these bounds with Theorem 2.2.1. $\square$

CHAPTER 3

Applications

3.1. The Couveignes-Edixhoven-Bruin algorithm

Let $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ be a congruence subgroup, and let k be a positive integer. Let

$$d(\Gamma) = [\mathrm{SL}_2(\mathbf{Z}) : \{\pm 1\}\Gamma]/12.$$

A modular form f of weight k for the group Γ is determined by k and its q-expansion coefficients $a_i(f)$ for $0 \leq i \leq k \cdot d(\Gamma)$; see [19] for definitions. In this section we follow [11] and give an algorithmic application of the main result of this thesis. More precisely, the goal of this section is to complete the proof of the following theorem. The proof is given at the end of this section.

Theorem 3.1.1. (Couveignes-Edixhoven-Bruin) *Assume the Riemann hypothesis for ζ-functions of number fields. Then there exists a probabilistic algorithm that, given*

- *a positive integer k,*
- *a number field K,*
- *a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$,*
- *a modular form f of weight k for Γ over K, and*
- *a positive integer m in factored form,*

computes $a_m(f)$ of f, and whose expected running time is bounded by a polynomial in the length of the input.

To make the above theorem into a precise statement, we explain in the following remarks how the number field K and the modular form f are supposed to be given.

Remark 3.1.2. In the algorithm, we represent K by its multiplication table with respect to some $\mathbf{Q}$-basis $(b_1, \ldots, b_r)$ of K. This means that we represent K by the rational numbers c_{ijk} with $1 \leq i, j, k \leq r$ such that

$$b_i b_j = \sum_{k=1}^{r} c_{ijk} b_k.$$

We represent elements of K as $\mathbf{Q}$-linear combinations of $(b_1, \ldots, b_r)$.

Remark 3.1.3. Let us make precise how the modular form f of weight k for the group Γ should be given. Firstly, a modular form f of weight k for the group Γ is determined by k and its q-expansion coefficients $a_i(f)$ for $0 \leq i \leq k \cdot d(\Gamma)$. For the algorithm, we represent f by its coefficients

$$a_0(f), \ldots, a_{k \cdot d(\Gamma)}(f).$$

These are all elements in K and are thus given as explained in Remark 3.1.2.

Remark 3.1.4. We have made precise how the number field K and the modular form f should be given to the algorithm, and how the algorithm returns the coefficient $a_m(f)$. We should also explain what "probabilistic" means in this context. The correct interpretation is that the result is guaranteed to be correct, but that the running time depends on random choices made during execution. We refer to [38] for a discussion of such probabilistic algorithms.

The above remarks make Theorem 3.1 into a precise mathematical statement.

Remark 3.1.5. The algorithm is due to Bruin, Couveignes and Edixhoven. Assume the Riemann hypothesis for ζ-functions of number fields. It was shown that the algorithm runs in polynomial time for **certain** congruence subgroups; see [11, Theorem 1.1]. Bruin did not have enough information about the semi-stable bad reduction of the modular curve $X_1(n)$ at primes p such that p^2 divides n to show that the algorithm runs in polynomial time. Nevertheless, our bounds on the discriminant of a curve can be used to show that the algorithm runs in polynomial time for **all** congruence subgroups.

Proof. We follow Bruin's strategy; see [10, Chapter V.1, p. 165]. In fact, Bruin notes that the algorithm runs in polynomial time for all congruence subgroups if, for all positive integers n, the discriminant $\Delta(X_1(n))$ is bounded by a polynomial in n. Now, by Theorem 2.1.1, the inequality $\Delta(X_1(n)) \leq 5 \cdot 10^8 \deg_B(X_1(n))^7$ holds. Note that

$$\deg_B(X_1(n)) \leq [\mathrm{SL}_2(\mathbf{Z}) : \Gamma_1(n)].$$

Since

$$[\mathrm{SL}_2(\mathbf{Z}) : \Gamma_1(n)] = n^2 \prod_{p|n}(1 - 1/p^2) \leq n^2,$$

we conclude that $\Delta(X_1(n)) \leq 5 \cdot 10^8 n^{14}$. We conclude that $\Delta(X_1(n))$ is bounded by a polynomial in n. □

3.2. Modular curves, Fermat curves, Hurwitz curves, Galois Belyi curves

Let X be a smooth projective connected curve over $\overline{\mathbf{Q}}$ of genus $g \geq 2$. We say that X is a Fermat curve if there exists an integer n such that X is isomorphic to the curve given by the equation $x^n + y^n = z^n$ in $\mathbf{P}^2_{\overline{\mathbf{Q}}}$. Moreover, we say that X is a Hurwitz curve if $\#\mathrm{Aut}(X) = 84(g-1)$. Also, we say that X is a Galois Belyi curve if the quotient $X/\mathrm{Aut}(X)$ is isomorphic to $\mathbf{P}^1_{\overline{\mathbf{Q}}}$ and the morphism $X \to X/\mathrm{Aut}(X)$ is ramified over exactly three points; see [12, Proposition 2.4], [66] or [67]. Note that Fermat curves and Hurwitz curves are Galois Belyi curves. Finally, we say that X is a modular curve if $X_{\mathbf{C}}$ is a classical congruence modular curve for some (hence any) embedding $\overline{\mathbf{Q}} \subset \mathbf{C}$.

If X is a Galois Belyi curve, we have $\deg_B(X) \leq 84(g-1)$. This follows from the Hurwitz bound $\#\mathrm{Aut}(X) \leq 84(g-1)$. In particular, by Proposition 1.9.12, there are only finitely many isomorphism classes of Galois Belyi curves of bounded genus.

Proposition 3.2.1. *If X is a modular curve, then* $\deg_B(X) \leq 128(g+1)$.

Proof. This result is due to Zograf; see [70]. Zograf's proof uses methods from the spectral theory of the Laplacian. Let us explain Zograf's proof more precisely. Let Γ be a cofinite Fuchsian group, i.e., a discrete subgroup of $\mathrm{SL}_2(\mathbf{R})$ such that the volume $\mathrm{vol}(\Gamma\backslash\mathbf{H})$ of $\Gamma\backslash\mathbf{H}$ is finite; see Section 2.3.3. (Recall that the hyperbolic metric μ_{hyp} on $\mathbf{H}$ induces a measure on $\Gamma\backslash\mathbf{H}$, and we take the volume of $\Gamma\backslash\mathbf{H}$ with respect to this measure.) The Laplace operator Δ on the space of smooth Γ-invariant functions on $\mathbf{H}$ with compact support (as a function on $\Gamma\backslash\mathbf{H}$) is defined as

$$\Delta = -y^2(\partial_x^2 + \partial_y^2),$$

where we write $\tau = x + iy$ on $\mathbf{H}$. This operator can be extended to an (unbounded) self-adjoint operator on the Hilbert space $\mathrm{L}^2(\Gamma\backslash\mathbf{H})$ of square-integrable complex-valued functions on $\mathbf{H}$ (with respect to the measure induced by μ_{hyp}), defined on a dense open subspace; we denote this extension by Δ as well. The spectrum of Δ consists of a discrete part and a continuous part. The discrete spectrum of Δ consists of eigenvalues of Δ and is of the form $\{\lambda_j\}_{j=0}^{\infty}$ with

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots, \quad \lambda_j \to \infty \text{ as } j \to \infty.$$

Zograf proves a generalization of the Yang-Yau inequality of a compact Riemann surface; see [68]. More precisely, Zograf proves that, if $g(\Gamma)$ denotes the genus of the compactification X_Γ of $\Gamma\backslash\mathbf{H}$ obtained by adding the cusps and we have $\mathrm{vol}(\Gamma\backslash\mathbf{H}) > 32\pi(g(\Gamma)+1)$, the inequality

$$\lambda_1 < \frac{8\pi(g(\Gamma)+1)}{\mathrm{vol}(\Gamma\backslash\mathbf{H})}$$

holds; see [70, Theorem 1]. This statement implies that, for Γ a congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$, the inequality

$$[\mathrm{SL}_2(\mathbf{Z}) : \Gamma] < 128(g(\Gamma) + 1)$$

holds, where $[\mathrm{SL}_2(\mathbf{Z}) : \Gamma]$ denotes the index of Γ in $\mathrm{SL}_2(\mathbf{Z})$; see [70, Corollary 1]. To prove the latter inequality, we argue by contradiction. In fact, suppose that

$$[\mathrm{SL}_2(\mathbf{Z}) : \Gamma] \geq 128(g(\Gamma) + 1).$$

Since $3\mathrm{vol}(\Gamma\backslash\mathbf{H}) = \pi[\mathrm{SL}_2(\mathbf{Z}) : \Gamma]$, we deduce that

$$\mathrm{vol}(\Gamma\backslash\mathbf{H}) \geq \frac{128}{3}\pi(g(\Gamma) + 1) > 32\pi(g(\Gamma) + 1).$$

Now, Zograf's generalization of the Yang-Yau inequality implies that

$$\lambda_1 < \frac{8\pi(g(\Gamma) + 1)}{\mathrm{vol}(\Gamma\backslash\mathbf{H})} \leq \frac{3}{16}.$$

That is, the smallest positive eigenvalue is strictly less than $3/16$. But this contradicts Selberg's famous lower bound for the smallest positive eigenvalue of the Laplace operator associated to a congruence subgroup, i.e., the first eigenvalue λ_1 of Δ is not less than $3/16$; see [55]. Finally, to deduce the upper bound for the Belyi degree of the congruence modular curve X_Γ, note that $\deg_B(X_\Gamma) \leq [\mathrm{SL}_2(\mathbf{Z}) : \Gamma]$. □

Corollary 3.2.2. *Let X be a smooth projective connected curve over $\overline{\mathbf{Q}}$ of genus $g \geq 1$. Suppose that X is a modular curve or Galois Belyi curve. Then*

$$\max(h_{\mathrm{Fal}}(X), e(X), \Delta(X), |\delta_{\mathrm{Fal}}(X)|) \leq 2 \cdot 10^{19} g^2 (g + 1)^5.$$

Proof. For X a modular curve, this follows from Theorem 2.1.1 and the inequality

$$\deg_B(X) \leq 128(g + 1)$$

due to Zograf (Proposition 3.2.1). For X a Galois Belyi curve, this follows from Theorem 2.1.1 and the inequality $\deg_B(X) \leq 84(g - 1)$. □

Remark 3.2.3. Let $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ be a finite index subgroup, and let X be the compactification of $\Gamma\backslash\mathbf{H}$ obtained by adding the cusps, where Γ acts on the complex upper half-plane $\mathbf{H}$ via Möbius transformations. Let $X(1)$ denote the compactification of $\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}$. The inclusion $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ induces a morphism $X \to X(1)$. There is a unique finite morphism $Y \to \mathbf{P}^1_{\overline{\mathbf{Q}}}$ of smooth projective connected curves over $\overline{\mathbf{Q}}$ corresponding to $X \longrightarrow X(1)$. The Belyi degree of Y is bounded from above by the index d of Γ in $\mathrm{SL}_2(\mathbf{Z})$. In particular, the inequality

$$\max(h_{\mathrm{Fal}}(Y), e(Y), \Delta(Y), |\delta_{\mathrm{Fal}}(Y)|) \leq 10^9 d^7$$

holds.

Remark 3.2.4. Non-explicit versions of Corollary 3.2.2 were previously known for certain modular curves. Firstly, polynomial bounds for Arakelov invariants of $X_0(n)$ with n squarefree were previously known; see [65, Théorème 1.1], [65, Corollaire 1.3], [2], [46, Théorème 1.1] and [34]. The proofs of these results rely on the theory of modular curves. Also, similar results for Arakelov invariants of $X_1(n)$ with n squarefree were shown in [20] and [44]. Finally, bounds for the self-intersection of the dualizing sheaf of a Fermat curve of prime exponent are given in [13] and [36].

3.3. Heights of covers of curves with fixed branch locus

For any finite subset $B \subset \mathbf{P}^1(\overline{\mathbf{Q}})$ and integer $d \geq 1$, the set of smooth projective connected curves X over $\overline{\mathbf{Q}}$ such that there exists a finite morphism $X \to \mathbf{P}^1_{\overline{\mathbf{Q}}}$ étale over $\mathbf{P}^1_{\overline{\mathbf{Q}}} - B$ of degree d is finite. In particular, the Faltings height of X is bounded by a real number depending only on B and d. In this section we prove an explicit version of the latter statement. To state our result we need to define the height of B.

For any finite set $B \subset \mathbf{P}^1(\overline{\mathbf{Q}})$, define the (exponential) height as

$$H_B = \max\{H(\alpha) : \alpha \in B\},$$

where the height $H(\alpha)$ of an element α in $\overline{\mathbf{Q}}$ is defined as

$$H(\alpha) = \left(\prod_v \max(1, \|\alpha\|_v)\right)^{1/[K:\mathbf{Q}]}.$$

Here K is a number field containing α and the product runs over the set of normalized valuations v of K. (As in [35, Section 2] we require our normalization to be such that the product formula holds.)

Theorem 3.3.1. *Let U be a non-empty open subscheme in $\mathbf{P}^1_{\overline{\mathbf{Q}}}$ with complement $B \subset \mathbf{P}^1(\overline{\mathbf{Q}})$. Let N be the number of elements in the orbit of B under the action of* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. *Define*

$$c(B) := (4NH_B)^{45N^3 2^{N-2} N!}.$$

Then, for any finite morphism $\pi : Y \to \mathbf{P}^1_{\overline{\mathbf{Q}}}$ étale over U, where Y is a smooth projective connected curve over $\overline{\mathbf{Q}}$ of genus $g \geq 1$,

$$\begin{array}{rcccl} -\frac{1}{2}\log(2\pi)g & \leq & h_{\mathrm{Fal}}(Y) & \leq & 13 \cdot 10^6 g c(B)(\deg \pi)^5 \\ 0 & \leq & e(Y) & \leq & 3 \cdot 10^7 (g-1) c(B)(\deg \pi)^5 \\ 0 & \leq & \Delta(Y) & \leq & 5 \cdot 10^8 g^2 c(B)(\deg \pi)^5 \\ -10^8 g^2 c(B)(\deg \pi)^5 & \leq & \delta_{\mathrm{Fal}}(Y) & \leq & 2 \cdot 10^8 g c(B)(\deg \pi)^5. \end{array}$$

Theorem 3.3.1 is a consequence of Theorem 3.3.3 given below where we consider branched covers of any curve over $\overline{\mathbf{Q}}$ (i.e., not only $\mathbf{P}^1_{\overline{\mathbf{Q}}}$). Firstly, let us explain how Theorem 3.3.1 settles a conjecture of Edixhoven-de Jong-Schepers ([22, Conjecture 5.1]). Let us start by stating their conjecture.

Conjecture 3.3.2. (Edixhoven-de Jong-Schepers) *Let $U \subset \mathbf{P}^1_{\mathbf{Z}}$ be a non-empty open subscheme. Then there are integers a and b with the following property. For any prime number ℓ, and for any connected finite étale cover $\pi : V \to U_{\mathbf{Z}[1/\ell]}$, the Faltings height of the normalization of $\mathbf{P}^1_{\mathbf{Q}}$ in the function field of V is bounded by $(\deg \pi)^a \ell^b$.*

Proof of Conjecture 3.3.2. We claim that this conjecture holds with $b = 0$ and an integer a depending only on $U_{\mathbf{Q}}$. In fact, let $U \subset \mathbf{P}^1_{\mathbf{Z}}$ be a non-empty open subscheme, and let $V \to U$ be a connected finite étale cover. Let $\pi : Y \to \mathbf{P}^1_{\mathbf{Q}}$ be the normalization of $\mathbf{P}^1_{\mathbf{Q}}$ in the function field of V and note that π is étale over $U_{\mathbf{Q}}$. Let $B = \mathbf{P}^1_{\mathbf{Q}} - U_{\mathbf{Q}} \subset \mathbf{P}^1(\overline{\mathbf{Q}})$ and let N be the number of elements in the orbit of B under the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. By Theorem 3.3.1,

$$h_{\mathrm{Fal}}(Y) := \sum_{X \subset Y_{\overline{\mathbf{Q}}}} h_{\mathrm{Fal}}(X) \leq (\deg \pi)^a,$$

where the sum runs over all connected components X of $Y_{\overline{\mathbf{Q}}} := Y \times_{\mathbf{Q}} \overline{\mathbf{Q}}$, and

$$a = 6 + \log\left(13 \cdot 10^6 N (4NH_B)^{45N^3 2^{N-2} N!}\right).$$

Here we used that, $g \leq N \deg \pi$ and

$$13 \cdot 10^6 g (4NH_B)^{45N^3 2^{N-2} N!} \leq (\deg \pi)^{1+\log\left(13 \cdot 10^6 N (4NH_B)^{45N^3 2^{N-2} N!}\right)}.$$

This proves Conjecture 3.3.2. □

We now state and prove Theorem 3.3.3. Let X be a smooth projective connected curve over $\overline{\mathbf{Q}}$. We prove that Arakelov invariants of (possibly ramified) covers of X are polynomial in the degree. Let us be more precise.

Theorem 3.3.3. *Let X be a smooth projective connected curve over $\overline{\mathbf{Q}}$, let U be a non-empty open subscheme of X, let $B_f \subset \mathbf{P}^1(\overline{\mathbf{Q}})$ be a finite set, and let $f : X \to \mathbf{P}^1_{\overline{\mathbf{Q}}}$ be a finite morphism of degree n unramified over $\mathbf{P}^1_{\overline{\mathbf{Q}}} - B_f$. Define $B := f(X - U) \cup B_f$. Let N be the number of elements in the orbit of B under the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ and let H_B be the height of B. Define*

$$c_B := (4NH_B)^{45N^3 2^{N-2} N!}.$$

Then, for any finite morphism $\pi : Y \to X$ étale over U, where Y is a smooth projective connected curve over $\overline{\mathbf{Q}}$ of genus $g \geq 1$,

$$\begin{array}{rcccl} -\frac{1}{2}\log(2\pi)g & \leq & h_{\mathrm{Fal}}(Y) & \leq & 13 \cdot 10^6 g c_B \cdot n^5 (\deg \pi)^5 \\ 0 & \leq & e(Y) & \leq & 3 \cdot 10^7 (g-1) c_B \cdot n^5 (\deg \pi)^5 \\ 0 & \leq & \Delta(Y) & \leq & 5 \cdot 10^8 g^2 c_B \cdot n^5 (\deg \pi)^5 \\ -10^8 g^2 c_B \cdot n^5 (\deg \pi)^5 & \leq & \delta_{\mathrm{Fal}}(Y) & \leq & 2 \cdot 10^8 g c_B \cdot n^5 (\deg \pi)^5. \end{array}$$

Proof. We apply Khadjavi's effective version of Belyi's theorem. More precisely, by [35, Theorem 1.1.c], there exists a finite morphism

$$R : \mathbf{P}^1_{\overline{\mathbf{Q}}} \to \mathbf{P}^1_{\overline{\mathbf{Q}}}$$

étale over $\mathbf{P}^1_{\overline{\mathbf{Q}}} - \{0, 1, \infty\}$ such that $R(B) \subset \{0, 1, \infty\}$ and

$$\deg R \leq (4NH_B)^{9N^3 2^{N-2} N!}.$$

Note that the composed morphism

$$R \circ f \circ \pi : Y \xrightarrow{\quad \pi \quad} X \xrightarrow{f} \mathbf{P}^1_{\overline{\mathbf{Q}}} \xrightarrow{R} \mathbf{P}^1_{\overline{\mathbf{Q}}}$$

is unramified over $\mathbf{P}^1_{\overline{\mathbf{Q}}} - \{0, 1, \infty\}$. We conclude by applying Theorem 2.1.1 to the composition $R \circ f \circ \pi$. □

Proof of Theorem 3.3.1. We apply Theorem 3.3.3 with $X = \mathbf{P}^1_{\overline{\mathbf{Q}}}$, B_f the empty set, and $f : X \to \mathbf{P}^1_{\overline{\mathbf{Q}}}$ the identity map. □

In the proof of Theorem 3.3.3, we applied Khadjavi's effective version of Belyi's theorem. Khadjavi's bounds are not optimal; see [40, Lemme 4.1] and [35, Theorem 1.1.b] for better bounds when B is contained in $\mathbf{P}^1(\mathbf{Q})$. Actually, the use of Belyi's theorem makes the dependence on the branch locus enormous in Theorem 3.3.3. It should be possible to avoid the use of Belyi's theorem and improve the dependence on the branch locus in Theorem 3.3.3.

Remark 3.3.4. We mention the quantitative Riemann existence theorem due to Bilu and Strambi; see [6]. Bilu and Strambi give explicit bounds for the naive logarithmic height of a cover of $\mathbf{P}^1_{\overline{\mathbf{Q}}}$ with fixed branch locus. Although their bound on the naive height is exponential in the degree, the dependence on the height of the branch locus in their result is logarithmic.

CHAPTER 4

Diophantine applications

In [23] Faltings proved the Shafarevich conjecture. That is, for a number field K, finite set S of finite places of K, and integer $g \geq 2$, there are only finitely many K-isomorphism classes of curves over K of genus g with good reduction outside S. This is a qualitative statement, i.e., this statement does not give an explicit bound on the "complexity" of such a curve.

In this chapter we are interested in quantitative versions of the Shafarevich conjecture, e.g., the effective Shafarevich conjecture and Szpiro's small points conjecture.

Our main result (joint with Rafael von Känel) is a proof of Szpiro's small points conjecture for cyclic covers of the projective line of prime degree; see Theorem 4.4.1. To explain a part of our proof, we have also found it fit to discuss the proof of the effective Shafarevich conjecture for cyclic covers of the projective line of prime degree due to de Jong-Rémond and von Känel in Section 4.2.1. We finish this chapter with a discussion of a result of Levin which gives some hope for obtaining applications of the results in this chapter to long-standing conjectures in Diophantine geometry.

The results of this chapter form only a small part of our article with von Känel [31]. In *loc. cit* we also discuss the optimality of the constant, and we give better bounds than those presented here.

4.1. The effective Shafarevich conjecture

In this section we follow Rémond ([51]). Firstly, we recall Faltings' finiteness theorem for abelian varieties.

Theorem 4.1.1. (Faltings [23]) *Let K be a number field, S a finite set of finite places of K and g an integer. Then there are only finitely many K-isomorphism classes of g-dimensional abelian varieties over K with good reduction outside S.*

An application of Torelli's theorem allows one to deduce the following finiteness theorem for curves from Theorem 4.1.1.

Theorem 4.1.2. *Let K be a number field, S a finite set of finite places of K and $g \geq 2$ an integer. Then there are only finitely many K-isomorphism classes of genus g curves over K with good reduction outside S.*

We are interested in an effective version of Faltings' finiteness theorem for curves. Let us consider the "effective Shafarevich" conjecture as stated in [51].

Conjecture 4.1.3. (Effective Shafarevich for curves) *Let K be a number field, S a finite set of finite places of K and $g \geq 2$ an integer. Then, there exists an explicit real number c (depending only on K, S and g) such that, for a smooth projective geometrically connected curve X of genus g over K with good reduction outside S,*

$$h_{\mathrm{Fal,stable}}(X) \leq c.$$

Remark 4.1.4. Removing the word "explicit" from Conjecture 4.1.3 gives a statement equivalent to Faltings' finiteness theorem for curves (Theorem 4.1.2). In fact, it is clear that such a statement follows from Faltings' finiteness theorem. Conversely, the above conjecture (with or without the word "explicit") implies that, for any number field K, finite set of finite places S of K and integer $g \geq 2$, there are only finitely many K-isomorphism classes of genus g curves over K with semi-stable reduction over O_K and good reduction outside S. Here we use the "Northcott property" of the Faltings height (Theorem 1.6.5). To obtain Theorem 4.1.2, we argue as follows. For a curve X over K of genus $g \geq 2$ with good reduction outside S, there exists a field extension L/K of bounded degree in g and ramified only over $S \cup \{3, 5\}$ such that X_L has semi-stable reduction over O_L. Thus, by the Hermite-Minkowski theorem, it suffices to show that, for a finite Galois extension L/K and smooth projective geometrically connected curve X of genus at least two over K, there are only finitely many curves X' over K such that X'_L is isomorphic to X_L. Note that the set of such X' is in one-to-one correspondence with $H^1(\mathrm{Gal}(L/K), \mathrm{Aut}_{\overline{K}}(X_{\overline{K}}))$. Since $\mathrm{Aut}_{\overline{K}}(X_{\overline{K}})$ is finite, the cohomology set

$$H^1(\mathrm{Gal}(L/K), \mathrm{Aut}_{\overline{K}}(X_{\overline{K}}))$$

is finite. This proves Theorem 4.1.2.

4.2. The effective Shafarevich conjecture for cyclic covers

In this section we follow de Jong-Rémond ([17]).

For a number field K, let $\Delta = |\Delta_{K/\mathbf{Q}}|$ be its absolute discriminant. For a finite set of finite places S of a number field K, let

$$\Delta_S = \Delta \exp\left((\sum_{\mathfrak{p}\in S} \log N_{K/\mathbf{Q}}(\mathfrak{p}) + [K:\mathbf{Q}]\log 4)^2 \right).$$

The following theorem is the main result of *loc. cit.* and proves Conjecture 4.1.3 for cyclic covers of the projective line of prime degree.

Theorem 4.2.1. (de Jong-Rémond) *Let K be a number field, S a finite set of finite places of K and g an integer. Let X be a smooth projective geometrically connected curve of genus g over K with good reduction outside S. Suppose that there exists a finite morphism $X \to \mathbf{P}^1_K$ such that $X_{\overline{K}} \to \mathbf{P}^1_{\overline{K}}$ is a cyclic cover of prime degree for some (hence any) algebraic closure $K \to \overline{K}$. Then*

$$h_{\mathrm{Fal,stable}}(X) \leq 2^{2^{22}g^g} \Delta_S^{2^{15}g^5}.$$

In this section we aim at explaining the main ingredients of the proof of Theorem 4.2.1. The proof of de Jong-Rémond is obtained in five steps which we will give below. We will give the proof of Theorem 4.2.1 at the end of this section. The first step is to replace the Faltings height of X by the theta height $h_\theta(X)$ of the Jacobian of X with respect to its principal polarization induced by the theta divisor; see [50, Definition 2.6] or [51, Section 4.a].

Lemma 4.2.2. (Bost-David-Pazuki) *Let $g \geq 1$ be an integer. Then, for a smooth projective geometrically connected genus g curve X over $\overline{\mathbf{Q}}$, the inequality*

$$h_{\mathrm{Fal}}(X) \leq 2h_\theta(X) + 2^{5g+1}\,(2 + \max(1, h_\theta(X))$$

holds.

Proof. This follows from [50, Corollary 1.3]. (Note that we are working with $r = 4$ here in the notation of *loc. cit.*.) □

The second step consists of invoking an explicit upper bound for the theta height due to Rémond ([52]). Let K be a number field, $K \to \overline{K}$ an algebraic closure of K, S a finite set of finite places and g an integer. Let X be a smooth projective geometrically connected curve over K. Let $X \to \mathbf{P}^1_K$ be a finite morphism such that $X_{\overline{K}} \to \mathbf{P}^1_{\overline{K}}$ is a cyclic cover of prime degree. Let H be the height of the finite set of cross-ratios associated to the branch points of $X_{\overline{K}} \to \mathbf{P}^1_{\overline{K}}$; see Section 3.3 for the definition of the height of a finite set of algebraic numbers and [17] for the definition of the set of cross-ratios.

Lemma 4.2.3. *We have*

$$h_\theta(X_{\overline{K}}) \leq 2^{3360 \cdot g^3 8^g} H.$$

Proof. The computation can be found in [17, p. 1141-1142]. □

Thus, to prove Theorem 4.2.1, it suffices to bound H explicitly in terms of K, S and g. The idea is to show that every cross-ratio satisfies a well-studied Diophantine equation.

Lemma 4.2.4. (de Jong-Rémond) *Let b be a cross-ratio of the branch locus of $X_{\overline{K}} \to \mathbf{P}^1_{\overline{K}}$. Then, if $L = K(b)$ and $S' = S_L$, we have that b and $1-b$ are S_L-units in L.*

Proof. By applying [17, Proposition 2.1] to b, $1-b$, b^{-1} and $(1-b)^{-1}$, it follows that b, $1-b$, b^{-1} and $(1-b)^{-1}$ are S_L-integers in L. This implies that b and $1-b$ are S_L-units in L. □

The fourth step consists of applying the well-established theory of logarithmic forms ([4]).

Lemma 4.2.5. (Baker-Győry-Yu) *Let L be a number field and S_L a finite set of finite places of L. Let d, R and P be the degree of L over $\mathbf{Q}$, the regulator of L over $\mathbf{Q}$ and the maximum of $|N_{L/\mathbf{Q}}(\mathfrak{p})|$ as $\mathfrak{p}$ runs over S_L. Let $s = \#S_L + d$. Then, if b and $1-b$ are S_L-units in L, the inequality*

$$h(b) \leq 2^{15}(16sd)^{2s+4} PR \left(1 + \frac{\max(1, \log R)}{\max(1, \log P)}\right)$$

holds.

Proof. This is an application of the main result of [28] to b. In fact, the pair $(b, 1-b)$ is a solution of the equation $x + y = 1$ with $(x, y) \in \mathcal{O}^\times_{S_L} \times \mathcal{O}^\times_{S_L}$. (See the proof of [15, Lemme 3.1] for details.) □

The preceding two lemmata can be combined into giving an explicit upper bound for H.

Lemma 4.2.6. *We have*

$$H \leq \Delta_S^{(8g)^5}.$$

Proof. Every cross-ratio is an S_L-unit, and by Lemma 4.2.5, the height of such an algebraic number can explicitly bounded in terms of the degree $[L : \mathbf{Q}]$, the regulator of L over $\mathbf{Q}$ and the maximum of $|N_{L/\mathbf{Q}}(\mathfrak{p})|$ as $\mathfrak{p}$ runs over S_L. This explicit bound implies an explicit upper bound in terms of K, S and g. This computation requires some results from algebraic number theory; see [17, p. 1139-1140] for the proof. □

Proof of Theorem 4.2.1. By Lemma 4.2.2, it suffices to bound the theta height $h_\theta(X)$ explicitly (in terms of K, S and g). By Lemma 4.2.3, it suffices to bound H explicitly. This is precisely the content of Lemma 4.2.6. □

4.3. Szpiro's small points conjecture

We consider Szpiro's small points conjecture; see [60], [61], [63], [64], [59].

Conjecture 4.3.1. (Szpiro's small points conjecture) *Let K be a number field, $K \to \overline{K}$ an algebraic closure of K, S a finite set of finite places of K and $g \geq 2$ an integer. Then, there exists an explicit real number c such that, for a smooth projective geometrically connected curve X of genus g over K with good reduction outside S, there is a point a in $X(\overline{K})$ with*

$$h(a) \leq c.$$

A point a satisfying the conclusion of Conjecture 4.3.1 is called a "small point". Roughly speaking, the following theorem shows that the existence of a small point on X is equivalent to an explicit upper bound for $e(X)$.

Theorem 4.3.2. *Let X be a smooth projective connected curve over $\overline{\mathbf{Q}}$ of genus $g \geq 2$. Then, for all a in $X(\overline{\mathbf{Q}})$, the inequality*

$$e(X) \leq 4g(g-1)h(a)$$

holds. Moreover, for any $\epsilon > 0$, there exists a in $X(\overline{\mathbf{Q}})$ such that

$$h(a) \leq \frac{e(X)}{4(g-1)} + \epsilon.$$

Proof. The first statement is due to Faltings; see Theorem 2.2.1. The second statement follows from Faltings' Riemann-Roch theorem (Section 1.2) and is due to Moret-Bailly; see the proof of [48, Proposition 3.4]. □

Remark 4.3.3. Removing the word "explicit" from Conjecture 4.3.1 gives a statement equivalent to Faltings' finiteness theorem for curves (Theorem 4.1.2). In fact, the Arakelov invariant $e(X)$ satisfies the following Northcott property. Let C be a real number, and let $g \geq 2$ be an integer. For a number field K, there are only finitely many K-isomorphism classes of smooth projective connected curves X over K of genus g with semi-stable reduction over O_K and $e_{\text{stable}}(X) \leq C$. Thus, since $e(X) \leq 4g(g-1)h(b)$ for any b in $X(\overline{K})$, to deduce Faltings' finiteness theorem for curves from Conjecture 4.3.1, we can argue as in Remark 4.1.4.

4.4. Szpiro's small points conjecture for cyclic covers

The following theorem proves Szpiro's small points conjecture (Conjecture 4.3.1) for cyclic covers of the projective line of prime degree.

Theorem 4.4.1. **([31, Theorem 3.1])** *Let K be a number field of degree d over $\mathbf{Q}$, S a finite set of finite places of K and g an integer. Let X be a smooth projective geometrically connected curve of genus g over K with good reduction outside S. Suppose that there exists a finite morphism $\pi : X \to \mathbf{P}^1_K$ such that $\pi_{\overline{K}} : X_{\overline{K}} \to \mathbf{P}^1_{\overline{K}}$ is a cyclic cover of prime degree for some algebraic closure $K \to \overline{K}$. Then there exists a in $X(\overline{K})$ such that*

$$h(a) \leq \frac{10^7}{g}\left(4d!(2g+2)\Delta_S^{(8g)^5}\right)^{45(d!(2g+2))^3 2^{d!(2g+1)-2}(d!(2g+1))!}(2g+1)^5.$$

Proof. We may and do assume that 0, 1, and ∞ are branch points of the finite morphism $\pi : X \to \mathbf{P}^1_K$. Now, by Corollary 2.5.3, there exists a in $X(\overline{\mathbf{Q}})$ such that

$$h(a) \leq 10^7 \frac{\deg_B(X)^5}{g}.$$

To bound $\deg_B(X)$, we argue as in the proof Theorem 3.3.3. In fact, by Khadjavi's effective version of Belyi's theorem ([35, Theorem 1.1.c]), the inequality

$$\deg_B(X) \leq (4NH_B)^{9N^3 2^{N-2} N!} \deg \pi$$

holds, where B is the branch locus of $\pi_{\overline{K}}$, H_B is the height of the set B, and N is the number of elements in the orbit of B under the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Let H be the height of the finite set of cross-ratios associated to B. Note that

$$N \leq [K:\mathbf{Q}]!\#B, \quad \#B \leq 2g+2, \quad \deg \pi \leq 2g+1, \quad H_B \leq H,$$

where the first inequality is clear, the second inequality and third inequality follow from Riemann-Hurwitz, and the last inequality follows from the fact that every algebraic number α different from 0 and 1 equals the cross ratio of 0, 1, ∞ and α. By Lemma 4.2.6, $H \leq \Delta_S^{(8g)^5}$, where Δ_S is as in Section 4.2.1. Putting these inequalities together implies the theorem. □

Remark 4.4.2. The above proof of Theorem 4.4.1 gives a very large upper bound on $h(a)$. We actually give a much better upper bound for $h(a)$ in our article [31]. In fact, we prove that

$$h(a) \leq \exp\left(\mu^{d\mu}(N_S\Delta)^{\mu}\right),$$

where we let $d = [K : \mathbf{Q}]$, Δ the absolute discriminant of K over $\mathbf{Q}$, $N_S = \prod_{v\in S} N_v$, and $\mu = d(5g)^5$. In *loc. cit.* we also study the optimality of the upper bound, we study points with small Néron-Tate height, and we improve its value under further restrictive assumptions on X.

4.5. Zhang's lower bound for $e(X)$

In this section we prove a slightly stronger version of Szpiro's small points conjecture for cyclic covers of prime degree of the projective line.

Theorem 4.5.1. **([31, Theorem 3.1])** *Let K be a number field, S a finite set of finite places of K and g an integer. Let X be a smooth projective geometrically connected curve of genus g over K with good reduction outside S. Suppose that there exists a finite morphism $X \to \mathbf{P}^1_K$ such that $X_{\overline{K}} \to \mathbf{P}^1_{\overline{K}}$ is a cyclic cover of prime degree. Then there are infinitely many a in $X(\overline{K})$ with*

$$h(a) \leq 2 \cdot 10^7 \left(4d!(2g+2)\Delta_S^{(8g)^5}\right)^{45(d!(2g+2))^3 2^{d!(2g+1)-2}(d!(2g+1))!}(2g+1)^5.$$

To prove Theorem 4.5.1, we will apply the following result of Zhang.

Theorem 4.5.2. *There are infinitely many points a in $X(\overline{\mathbf{Q}})$ such that*

$$h(a) \leq \frac{e(X)}{2(g-1)}.$$

Proof. This follows from [69, Theorem 6.3]. □

Proof of Theorem 4.5.1. Theorem 4.5.1 is a consequence of Theorem 4.4.1 and the above result of Zhang. In fact, by Zhang's result and Faltings' inequality ([24, Theorem 5]), there are infinitely many points a in $X(\overline{\mathbf{Q}})$ such that, for all b in $X(\overline{\mathbf{Q}})$, the inequality

$$h(a) \leq 4g(g-1)e(X) \leq 2gh(b)$$

holds. □

4.6. Diophantine applications of the effective Shafarevich conjecture (after Levin)

In this section we follow Levin ([39]). Faltings proved the Mordell conjecture via the Shafarevich conjecture. In fact, in [49] Parshin famously proved that the Shafarevich conjecture for curves (Theorem 4.1.2) implies Mordell's conjecture.

Theorem 4.6.1. **(Faltings)** *For a number field K and smooth projective geometrically connected curve X over K of genus at least two, the set $X(K)$ of K-rational points on X is finite.*

Rémond proved that the effective Shafarevich conjecture (Conjecture 4.1.3) implies an "effective version of the Mordell conjecture". His proof relies on Kodaira's construction. For the sake of brevity, we only state a consequence of Rémond's result. We refer the reader to [51, Théorème 5.3] for a more precise statement.

Theorem 4.6.2. **([51, Théorème 5.3])** *Assume Conjecture 4.1.3. Let K be a number field and X a smooth projective geometrically connected curve over K of genus $g \geq 2$. Then there exists an explicit real number c such that, for all $a \in X(K)$, we have*

$$h(a) \leq c.$$

Remark 4.6.3. An explicit expression for c is given in [51, Théorème 5.3].

It is natural to ask whether "weak versions" of the effective Shafarevich conjecture have Diophantine applications. For instance, one could ask whether Theorem 4.2.1 implies "a weak effective version of the Mordell conjecture". Currently, no such implication is known. Nevertheless, it seems reasonable to suspect that some "weak version" of the effective Shafarevich conjecture implies some version of Siegel's theorem.

Theorem 4.6.4. (Siegel) *Let X be a smooth quasi-projective curve over a number field K, S a finite set of places of K, $O_{K,S}$ the ring of S-integers, and $f \in K(X)$. If X is a rational curve, then we assume further that f has at least three distinct poles. Then the set of S-integral points of X with respect to f,*

$$X(f, K, S) = \{a \in X(K) \mid f(a) \in O_{K,S}\}$$

is finite.

In general, there is no quantitative version of Siegel's theorem known, i.e., there is no known algorithm for explicitly computing the set $X(f, K, S)$. Of course, in some special cases there are known techniques for effectively computing $X(f, K, S)$; see [39, Section 1]. This ineffectivity arises in the classical proofs of Siegel' theorem from the use of Roth's theorem. (Actually, a weaker version of Roth's theorem due to Thue and Siegel is used.)

Theorem 4.6.5. (Roth [53]) *Let θ be a real algebraic number of degree $d \geq 2$. For all $\epsilon > 0$, there are only finitely many rational numbers p/q, with $p, q \in \mathbf{Z}$ coprime, such that*

$$|\theta - \frac{p}{q}| \leq \frac{1}{|q|^{2+\epsilon}}.$$

Currently, Roth's theorem remains ineffective. That is, if θ is a real algebraic number of degree $d \geq 2$, there is no known algorithm (in general!) for explicitly computing the set of rational numbers p/q such that $|\theta - p/q| \leq \frac{1}{|q|^{2+\epsilon}}$.

An interesting result of Levin shows that an effective version of the Shafarevich conjecture for hyperelliptic Jacobians has Diophantine applications. In fact, Levin proves that an effective Shafarevich conjecture for hyperelliptic Jacobians implies an effective version of Siegel's theorem for integral points on hyperelliptic curves. We interpret his result as to give some hope for obtaining applications of the results in this chapter to effective Diophantine conjectures such as Siegel's theorem.

Theorem 4.6.6. *([39, Theorem 3]) Let $g \geq 2$ be an integer. Suppose that, for any number field K and finite set of finite places S of K the set of K-isomorphism classes of hyperelliptic Jacobians*

$J = \mathrm{Jac}(C)$ over K of genus g with good reduction outside S is effectively computable (e.g., an explicit hyperelliptic Weierstrass equation for each such curve C is given). Then for any number field K, any finite set of places S of K, any hyperelliptic curve X over K of genus g, and any rational function f in $K(X)$, the set of S-integral points with respect to f,

$$X(f, K, S) = \{a \in X(K) \mid f(a) \in O_{K,S}\}$$

is effectively computable.

Levin's proof uses a slight variation on Parshin's proof of the well-known implication mentioned before "Shafarevich implies Mordell". It remains to be seen whether one can use Parshin-type constructions to obtain applications of the results in this chapter to effective Diophantine conjectures.

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Samenvatting

Zij $\mathbf{Q}$ het lichaam van rationale getallen en $\mathbf{Q} \to \overline{\mathbf{Q}}$ een algebraïsche afsluiting van $\mathbf{Q}$. In dit proefschrift bestuderen wij krommen over $\overline{\mathbf{Q}}$ en bewijzen wij ongelijkheden voor Arakelovinvarianten geassocieerd aan een kromme over $\overline{\mathbf{Q}}$.

Zij X een kromme over $\overline{\mathbf{Q}}$ van geslacht $g \geq 1$ met Belyigraad $\deg_B(X)$. Wij bewijzen in dit proefschrift dat de Faltingshoogte $h_{\mathrm{Fal}}(X)$ van X voldoet aan de ongelijkheid

$$h_{\mathrm{Fal}}(X) \leq 10^9 \deg_B(X)^7.$$

Met andere woorden, de Faltingshoogte van een kromme is polynomiaal begrensd in de Belyigraad. Wij laten ook zien dat de discriminant $\Delta(X)$ van X voldoet aan de ongelijkheid

$$\Delta(X) \leq 10^9 \deg_B(X)^7.$$

Deze twee ongelijkheden generalizeren wij als volgt. Zij $h(X)$ een Arakelovinvariant van X zoals gedefinieerd in het proefschrift. Dan geldt er dat

$$h(X) \leq 10^9 \deg_B(X)^7.$$

Het belang van expliciete ongelijkheden voor Arakelovinvarianten werd voor het eerst opgemerkt door Parshin in de jaren tachtig. Men kan laten zien dat een expliciete bovengrens voor de Faltingshoogte van een kromme over een getallenlichaam K, van gegeven geslacht en slechte reductie over de getallenring O_K van K, een effectieve versie van de stelling van Faltings (*quondam* Mordell's vermoeden) zou impliceren. In deze algemeenheid zijn dergelijke bovengrenzen niet bekend. Desalniettemin bewijzen wij in dit proefschrift dergelijke bovengrenzen voor cyclische overdekkingen van priemgraad van de projectieve lijn $\mathbf{P}^1_K$. Wij bewijzen hiermee een speciaal geval van Szpiro's *small points conjecture*.

Polynomiale ongelijkheden voor Arakelovinvarianten worden op een cruciale wijze toegepast in het werk van Peter Bruin, Jean-Marc Couveignes en Bas Edixhoven betreffende computationale aspecten van modulaire vormen en Galoisrepresentaties. De bovengenoemden vereisten polynomiale grenzen voor Arakelovinvarianten van modulaire krommen. Het eindproduct van dit proefschrift is een veralgemenisering van de ongelijkheden voor Arakelovinvarianten van modulaire krommen in termen van hun niveau.

Het is zeer aannemelijk dat de methodes van Bruin, Couveignes en Edixhoven kunnen worden gegeneralizeerd om de Galoisrepresentaties geassocieerd aan een oppervlak over $\mathbf{Q}$ te bepalen. Onze bijdrage aan dit probleem is een bewijs van een vermoeden van Edixhoven, de Jong en Schepers. Wij bewijzen dat, als X een kromme is over $\overline{\mathbf{Q}}$, B een eindige verzameling gesloten punten is op X en $Y \to X$ een overdekking is van graad d onvertakt over $X - B$, de ongelijkheid

$$h_{\mathrm{Fal}}(Y) \leq c(X, B) \cdot d^7$$

geldt, met $c(X, B)$ een reëel getal dat alleen afhangt van X en B. Dit resultaat zal hopelijk worden toegepast om te bewijzen dat er een polynomiaal algoritme is dat de étale cohomologie als Galoisrepresentatie van een oppervlak over $\mathbf{Q}$ bepaalt.

Résumé

Soient $\mathbf{Q}$ le corps des nombres rationnels et $\mathbf{Q} \to \overline{\mathbf{Q}}$ une clôture algébrique de $\mathbf{Q}$. Dans cette thèse nous considérons des courbes sur $\overline{\mathbf{Q}}$. Nous montrons des inégalités pour les invariants arakeloviens d'une courbe sur $\overline{\mathbf{Q}}$.

Soit X une courbe sur $\overline{\mathbf{Q}}$ de genre $g \geq 1$ et de degré de Belyi $\deg_B(X)$. Nous montrons dans cette thèse que la hauteur de Faltings $h_{\mathrm{Fal}}(X)$ de X satisfait l'inégalité

$$h_{\mathrm{Fal}}(X) \leq 10^9 \deg_B(X)^7.$$

C'est-à-dire, la hauteur de Faltings d'une courbe est bornée par un polynôme en le degré du Belyi. De plus, nous prouvons que le discriminant $\Delta(X)$ de X satisfait l'inégalité

$$\Delta(X) \leq 10^9 \deg_B(X)^7.$$

Nous généralisons les deux inégalités ci-dessus de la manière suivante. En effet, si $h(X)$ est un invariant arakeloviens associé à X (comme défini dans cette thèse), nous montrons que

$$h(X) \leq 10^9 \deg_B(X)^7.$$

Soient K un corps de nombres, $g \geq 2$ un entier et S un ensemble fini de places finies de K. Parshin a remarqué en premier l'importance d'une majoration explicite en K, g et S de l'invariant arakelovien $e(X)$ pour toutes les courbes X sur K de genre g et de bonne réduction en dehors de S. En effet, il a montré qu'un tel résultat impliquerait une version effective du théorème de Faltings (*quondam* la conjecture de Mordell). Malheureusement, il est très difficile de démontrer de telles inégalités. Dans cette thèse nous déduisons de nos inégalités citées ci-dessus un résultat plus faible que celui espéré par Parshin. En effet, nous montrons une majoration explicite pour $e(X)$ si X est un revêtement cyclique de la droite projective de degré premier. Nous démontrons en particulier la conjecture des petits points de Szpiro pour ces courbes.

Dans les travaux de Peter Bruin, Jean-Marc Couveignes et Bas Edixhoven sur le calcul de coefficients de formes modulaires et représentations galoisiennes, il s'avère important de borner des invariants arakeloviens des courbes modulaires $X_1(n)$ dans le niveau n. Le produit final de

cette thèse est une généralisation des inégalités pour les courbes modulaires obtenues par les mathématiciens mentionnés ci-dessus. En effet, en remplaçant la courbe modulaire $X_1(n)$ de niveau n par une courbe X quelconque définie sur $\overline{\mathbf{Q}}$ et en remplaçant le niveau n par le degré de Belyi de X, le résultat principal de cette thèse implique la généralisation énoncée précédemment.

Il semble possible que les méthodes de Bruin, Couveignes et Edixhoven puissent être généralisées pour calculer les représentations galoisiennes associé à une surface sur $\mathbf{Q}$. Notre contribution à ce problème est une démonstration d'une conjecture de Edixhoven, de Jong et Schepers sur la hauteur de Faltings d'un revêtement de la droite projective. En d'autres termes, nous démontrons le résultat suivant. Soient X une courbe sur $\overline{\mathbf{Q}}$, B un ensemble fini de points fermés de X et $Y \to X$ un revêtement fini de degré d qui est étale au-dessus de $X \backslash B$. Alors, on a l'inégalité

$$h_{\mathrm{Fal}}(Y) \leq c(X, B) \cdot d^7.$$

Ici $c(X, B)$ est un nombre réel qui dépend uniquement de X et de B. Nous espérons que ce résultat sera utile dans le calcul des représentations galoisiennes associées à une surface sur $\mathbf{Q}$.

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Curriculum Vitae

Ariyan Javan Peykar werd geboren op 15 december 1987 te Mashad. Hij groeide op in Zoetermeer en heeft daar in 2004 zijn HAVO diploma aan 't Alfrink college behaald. Na de voltooiing van zijn propedeuse aan de Technische Hogeschool Rijswijk begon hij natuurkunde en wiskunde te studeren in Leiden. Aanvankelijk streefde hij een loopbaan in de theoretische natuurkunde na, maar na het behalen van zijn dubbele propedeuse besloot hij zich te richten op de wiskunde. Hij vertrok na het behalen van zijn Bachelordiploma in 2008 naar Parijs om zijn loopbaan in de wiskunde te beginnen. In juli 2010 studeerde hij af op een scriptie getiteld *The Grothendieck-Riemann-Roch theorem*, onder begeleiding van prof. dr. Jaap Murre, in het kader van het *Algebra, Geometry and Number theory* programma.

Vervolgens ging Ariyan als promovendus aan de slag met een beurs van het ALGANT-consortium. Het proefschrift dat voor u ligt, is het resultaat van het onderzoek dat is gedaan onder begeleiding van prof. dr. Jean-Benoît Bost, prof. dr. Bas Edixhoven en dr. Robin de Jong aan l'Université de Paris-Sud 11 en de Universiteit Leiden.

Vanaf het najaar 2013 zal Ariyan als postdoctoraal onderzoeker verbonden zijn aan de Johannes Gutenberg-Universität in Mainz.