Seminar: Derived categories

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Organisation:
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The goal of this seminar is to understand the derived category associated to the category of coherent sheaves of a smooth projective variety. Especially, we would like to study under which assumptions two smooth projective varieties with equivalent derived categories are isomorphic. Our main reference is the book by Huybrechts [Hu].

Note that every two weeks Prof. Klaus will lecture on homological algebra on Thursday from 14h-15:30h. It might be useful for the participants of this seminar to attend (at least the first part) of his lecture.

Talk 1: Triangulated categories 1

Give the definition of additive and abelian categories. Introduce the Serre functor and triangulated categories. Prove that any Serre functor on a triangulated category is exact ([Hu, Proposition 1.46]). You can closely follow §1.1 and §1.2 in [Hu]. Alternatively, you can have a look at [GM] or [We].

Talk 2: Triangulated categories 2

Introduce the spanning classes and decompositions into triangulated subcategories. Further, give the definitions of exceptional objects and sequences and of semi-orthogonal decompositions. This talk should cover [Hu] §1.3 and §1.4. You can also have a look at [GM] or [We].

Talk 3: Derived categories

Give the construction of the derived category of an abelian category and show that it is indeed a triangulated category ([Hu, Proposition 2.24]. Here, you can follow [Hu] §2.1. See also §10 in [We] or [GM, Chapter 3.1–3.4] for more details.

Talk 4: Derived functors

Following [Hu] 2.2, explain the construction of the right derived functor associated to a left exact functor and discuss its properties. See also [GM, Chapter 3.6].

Talk 5: Spectral sequences

Recall shortly the notion of a spectral sequence and explain how spectral sequences occur whenever two derived functors are composed ([Hu, Proposition 2.66]). Further, give the definition of an ample sequence and show that they span the derived category. This talk should cover 2.3 and 2.4 in [Hu]. See also [GM, Chapter 3.7].

Talk 6: Derived categories of coherent sheaves

Define the bounded derived category of a scheme and discuss its properties, for instance that the Serre duality induces indeed a Serre functor ([Hu, Theorem 3.12]). Show, that the derived category $\mathcal{D}^b(X)$ is spanned by the $\kappa(x)$, $x \in X$ ([Hu, Propostion 3.17]) and by the powers of an ample line bundle ([Hu, Propostion 3.18]). This talk should cover §3.1 and §3.2 in [Hu]. To have an example, describe $\mathcal{D}^b(\mathbb{P}^n)$ in terms of generators by stating [Ca, Theorem 3.1].

Talk 7: Derived functors in algebraic geometry

Introduce all the derived functors in [Hu, §3.3] and discuss their properties. Explain the Grothendieck-Verdier duality ([Hu, Theorem 3.34]) .

Talk 8: The Bondal-Orlov reconstruction theorem

Prove that two smooth projective varieties with equivalent derived categories are isomorphic if the (anti-)canonical bundle of one of the varieties is ample ([Hu, Proposition 4.11]). Moreover, discuss the automorphism group of the derived category of a smooth projective variety with ample (anti-)canonical bundle ([Hu, Proposition 4.17]). Here, you can closely follow §4.1 and §4.2 in [Hu].

Talk 9: Fourier–Mukai transforms

Discuss Fourier–Mukai transforms as in [Hu, §5.1] and briefly sketch the proof of Orlov's theorem [Or, Theorem 2.2]. (Note that the proof is quite long.)

Talk 10: Abelian varieties

In the first part sketch the proof of the criterium for fully faithfulness [Hu, Proposition 7.1].

Next, prove that the derived category of an abelian variety and its dual are equivalent ([Hu, Proposition 9.19]). In particular, remark that there are non-isomorphic smooth projective varieties with equivalent derived categories ([Hu, Remark 9.22 (i)]). If time permits, discuss the relation between equivalences of the derived categories of two abelian varieties A and B and isomorphisms from $A \times \hat{A}$ to $B \times \hat{B}$ ([Hu, Proposition 9.39]).

Also, if time permits, show that for any abelian variety there are, up to isomorphisms, only finitely many abelian varieties with equivalent derived category ([Hu, Corollary 9.42]).

Talk 11: K3 surfaces

The Mukai pairing and Propositions 5.39 and 5.44 of [Hu] will be needed in this talk.

Recall the general facts about K3 surfaces as in [Hu, §10.1], especially the global Torelli theorem ([Hu, Theorem 10.4]). Then prove that two K3 surfaces have equivalent derived categories if and only if their Mukai lattices are isometric ([Hu, Proposition 10.10]). If time permits, explain the connection between Hodge isometries and equivalences of the derived categories ([Hu, Corollaries 10.12 and 10.13]).

References

[Ca] Caldararu, A.: *Derived categories of sheaves: a skimming.* arXiv:math/0501094

[Hu] Huybrechts, D.: *Fourier–Mukai transforms in Algebraic Geometry.*

[GM] Gelfand, S. I.; Manin, Y. I.: *Methods of Homological Algebra.*

[Or] Orlov, D.: *Equivalences of derived categories and K3 surfaces.* arXiv:alg-geom/9606006

[We] Weibel, C. A.: *An introduction to homological algebra.*