ARIYAN JAVANPEYKAR lec 3: "Arakelov Intersection Theory"

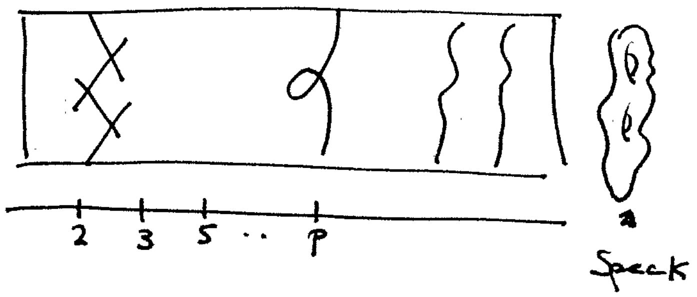
Motivation : $X/\overline{\mathbb{Q}}$ smooth proj connected curve
How to measure bad reduction of X?

Choose:
- a number field K, ring of integers $\mathcal{O}_K$
- an embedding $K \overset{\sigma}{\hookrightarrow} \overline{\mathbb{Q}}$
- a model $\mathcal{X}/\mathcal{O}_K$, i.e.

$$\begin{tikzcd}
\mathcal{X} \arrow[d] \arrow[dr, phantom, "\square"] & X_K \arrow[l] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[l] \arrow[d] \\
\operatorname{Spec} \mathcal{O}_K & \operatorname{Spec} K \arrow[l] & \operatorname{Spec} \overline{\mathbb{Q}} \arrow[l, "\sigma"']
\end{tikzcd}$$

with $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ proper flat
$\mathcal{X}$ integral regular
$\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ minimal semi-stable / nodal

This data always exists : Castelnuovo, Enriques, semi-st red. THM



Def $\delta_p := \# \operatorname{Sing}(\mathcal{X} \otimes \overline{k(p)})$ $p \in \operatorname{Spec} \mathcal{O}_K$ closed

(Rmk: all sing. points are ordinary double points)

$$\Delta(X) = \frac{1}{[K:\mathbb{Q}]} \sum_{\substack{p \subseteq \mathcal{O}_K \\ \text{max ideal}}} \delta_p \log(\# k(p)) \in \mathbb{R}_{\geq 0} \quad \text{well defined}$$

How to study $\Delta(X)$?

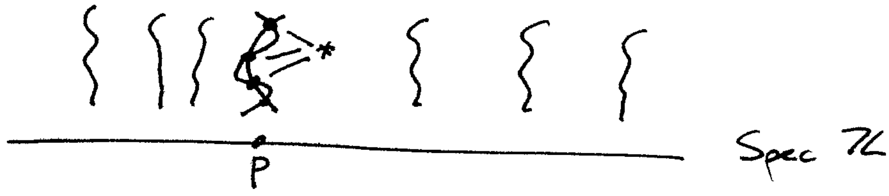
e.g. - $\Delta(X) = 0 \iff X$ has good reduction everywhere

- X genus 1 curve

Kodaira - Neron classif. : $I_n, [II, III, IV, I_n^*, II^*, III^*, IV^*]$

X genus two ... too many possibility for the reduction!

ex X modular curve, $X_1(p)$ p prime number



Deligne-Rapoport

$*$ (super singular ell. curves)

$$\Delta(X_1(p)) \leq p \log(p)$$

very difficult for higher level

We will use Arakelov's Intersection pairing on divisors on $\mathcal{X}$

<u>Thm</u> (J. 2014)

Let $X/\overline{\mathbb{Q}}$ be a smooth proj connected curve. Then

$$\Delta(X) \leq 10^9 \deg_B(X)^7$$

(COROLLARY: $\Delta(X_1(N)) \leq 10^9 N^{14}$)

INTRODUCTION TO ARAKELOV INTERSECTION PAIRING

B "curve" (irr. regular Noeth scheme, dim 1)

$\mathcal{X} \longrightarrow B$ "family over B" (flat proper morph. whose geom. fibers are connected curves $\mathcal{X}$ integral regular)

Ex ① $B = \mathbb{A}^1_{\mathbb{C}}$

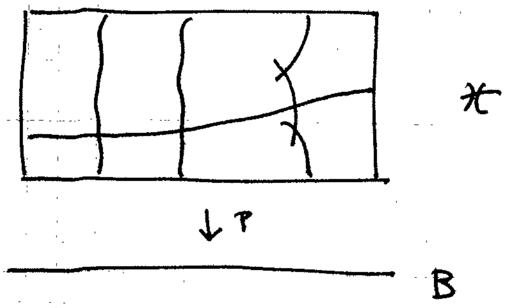
$$\begin{array}{c} \mathcal{X} = \mathbb{P}^1 \times \mathbb{A}^1_{\mathbb{C}} \\ \downarrow \\ B \end{array}$$

② $B = \operatorname{Spec} \mathbb{Z}$

$$\begin{array}{c} \mathcal{X} = \mathbb{P}^1_{\mathbb{Z}} \\ \downarrow \\ B \end{array}$$

Curves / Divisors on $\mathcal{X}$ (scheme of dim 2)

$\mathcal{D} \hookrightarrow \mathcal{X}$ " curve " i.e integral closed subscheme of cod. 1



$p(D) =$ closed irr in $B = \begin{cases} (b) & \text{VERTICAL} \\ B & \text{HORIZ.} \end{cases}$

(closed ← proper, irr ← irr)

every divisor $D = D^{hor} + D^{ver}$

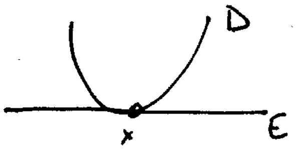
For $b \in B$ $\quad \mathrm{Div}_b(\mathcal{X}) = \{ \sum a_i D_i : \text{all } D_i \text{ vertical irred. } p(D_i) = b \}$

$\mathrm{Div}(\mathcal{X}) = \{ \text{divisors in } \mathcal{X} \} \supseteq \mathrm{Div}_b(\mathcal{X})$

$D, E \in \mathrm{Div}(\mathcal{X})$ with no common components

define $\quad i_x(D,E) = \mathrm{length}_{\mathcal{O}_{\mathcal{X},x}} \dfrac{\mathcal{O}_{\mathcal{X},x}}{\mathcal{O}_{\mathcal{X}}(-D)_x + \mathcal{O}_{\mathcal{X}}(-E)_x}$

Ex :



$y = x^2$

$y = 0$

$$\frac{k[x,y]}{(x^2 - y, y)} \cong \frac{k[x]}{x^2}$$

$$i_x(D,E) = 2$$

<u>Thm</u> (Liu's book, 9. 1.12)

Let B be either a curve or $\{b\}$, and $b \in B$.

There is a unique bilinear map

$$i_b : \mathrm{Div}(\mathcal{X}) \times \mathrm{Div}_b(\mathcal{X}) \longrightarrow \mathbb{Z} \quad \text{s.t.}$$

a) If D, E have no common components, then

$$i_b(D,E) = \sum_{x \in \mathcal{X}_b} i_x(D,E) \cdot [k(x) : k(b)]$$

b) $i_b \big|_{\mathrm{Div}_b(\mathcal{X}) \times \mathrm{Div}_b(\mathcal{X})}$ is symmetric

c) $i_b(D,E) = i_b(D',E)$ if $D \sim D'$

d) If $0 \leq E \subset \mathcal{X}_b$ $\quad i_b(D,E) = \deg(\mathcal{O}_{\mathcal{X}}(D))|_E$

<u>Cor</u>: If $B = \operatorname{Spec} k = \{pt\}$ then

$$\exists! \quad \begin{array}{ccc} \mathrm{Div}(\mathcal{X}) \times \mathrm{Div}(\mathcal{X}) & \longrightarrow & \mathbb{Z} \\ \downarrow & \circlearrowleft & \nearrow \exists! \\ \mathrm{Pic}(\mathcal{X}) \times \mathrm{Pic}(\mathcal{X}) & & \end{array}$$

pf: $\mathrm{Div}_b(\mathcal{X}) = \mathrm{Div}(\mathcal{X})$ use c) + b)

<u>Cor</u> $\mathrm{Pic}(B) = 0$ $\quad E \in \mathrm{Div}_b(\mathcal{X})$ $\quad E \cdot \mathcal{X}_b = 0$

<u>proof</u> $b \in B$ is principal (on B!) then $p^* b = \mathcal{X}_b$ is principal divisor on $\mathcal{X}$. Then use c) $\square$

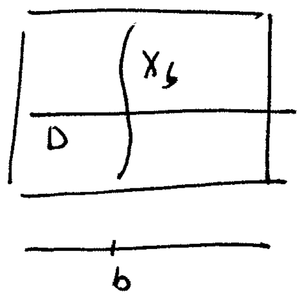
<u>Ex 1</u> $B = \mathbb{A}^1_{\mathbb{C}}$ $\quad b \in B$ closed pt

b is a principal divisor on $\mathbb{A}^1_{\mathbb{C}}$

$\rightarrow$ $\mathcal{X}_b$ is a principal divisor on $\mathcal{X}$

D horiz. divisor on $\mathcal{X}$. Then

$$D \cdot \mathcal{X}_b > 0$$



So intersection pairing <u>does NOT</u> respect linear equivalence on $\mathcal{X}$.

To remedy the situation

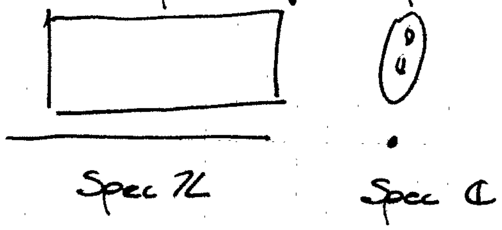
$$\begin{array}{ccc} \mathcal{X} & \hookrightarrow & \overline{\mathcal{X}} \\ \downarrow & & \downarrow \\ \mathbb{A}^1_{\mathbb{C}} & \hookrightarrow & \mathbb{P}^1_{\mathbb{C}} \end{array}$$

Ex (2) $\mathcal{B} = \operatorname{Spec} \mathbb{Z}$ $b \in \mathcal{B}$ closed point

$\mathcal{X}_b$ principal divisor D hor curve

$D \cdot \mathcal{X}_b > 0$

how to compactify ~~Spec Z~~ ?



difference is over Spec $\mathbb{C}$ line bundles can have metrics

solution : do ~~coute~~ at the same time arithm geometry / Spec $\mathbb{Z}$ & analytic geometry / Spec $\mathbb{C}$

§ ANALYTIC PART

X compact connected Riemann surf $g(X) \geq 1$

there is a natural hermitian inner product

$$(\omega, \eta) := \frac{i}{2} \int_X \omega \wedge \bar{\eta} \quad \text{on} \quad H^0(X, \Omega^1_X)$$

let $\omega_1 - \omega_g$ be an orthonormal basis

$$\mu_A = \mu = \frac{i}{2g} \sum_{i=1}^{g} \omega_i \wedge \overline{\omega}_i$$

1) μ is well def.
2) $\int_X \mu = 1$

Def

Arakelov Green Function G, is the unique function $X \times X \longrightarrow \mathbb{R}_{\geq 0}$

1) $G(p,q)^2$ is $\mathcal{C}^\infty$ on $X \times X$ and only vanishes along $\Delta \subset X \times X$

For all $p \in X$, For all $U_p \hookrightarrow X$ open with $p \in U_p$ $\forall z_p$ local coordinate at p

$$\log G(p,q) = \log |z_p(q)| + f(q) \quad \forall\ p \neq q \quad f \in \mathcal{C}^\infty$$

2) $\forall p \in X$ $p \neq q$ $\partial_q \bar{\partial}_q \log G(p,q) = 2\pi i \mu_{Ar}(q)$

3) For all $p \in X$

$$\int_X \log G(p,q)\, \mu(q) = 0$$

Ex: $G(p,q) = G(q,p)$ $\Rightarrow$ Ar intersect pairing is symmetric

§ ADMISSIBLE METRICS

$p \in X$ $\quad s \in \mathcal{O}_X(p)$ canonical generating section

$\|\cdot\|_{\mathcal{O}_X(p)}$ on $\mathcal{O}_X(p)$ to be the smooth hermitian metric

$\forall q \in X \quad \|s\|_{\mathcal{O}_X(p)}(q) = G(p,q)$

$\mathrm{curv}(\mathcal{O}_X(p), \|\cdot\|_{\mathcal{O}_X(p)}) = \mu$

$D \in \mathrm{Div}(\mathcal{X})$ $\quad \|\cdot\|_{\mathcal{O}_X(D)}$ via tensor products

$\mathrm{curv}(\mathcal{O}_X(D), \|\cdot\|_{\mathcal{O}_X(D)}) = (\deg D)\mu$

Def Let $(\mathcal{L}, \|\cdot\|)$ be a metrized line bundle. Then $(\mathcal{L}, \|\cdot\|)$ is admissible if $\mathrm{curv}(\mathcal{L}, \|\cdot\|)$ is a multiple of μ.

$\widehat{\mathrm{Pic}(\mathcal{X})} = \{ (\mathcal{L}, \|\cdot\|) \text{ admissible} \}$

$\to$ back to arithmetic

$$\begin{array}{l} \mathcal{X} \\ p \downarrow \\ \mathrm{Spec}\, \mathbb{Z} \end{array}$$

D Arakelov divisor on $\mathcal{X}$

$D = D_{fin} + D_{inf}$

where D_{fin} is Weil divisor on $\mathcal{X}$

$D_{inf} = \alpha\, [F_\infty]$ $\quad \alpha \in \mathbb{R}$

formal symbol "fiber at infinity"

$\widehat{\mathrm{Div}}(\mathcal{X})$ = Arakelov divisor on $\mathcal{X}$
$= \mathrm{Div}(\mathcal{X}) \oplus \mathbb{R}[F_\infty]$

$f \in K(\mathcal{X})^*$ $\quad \widehat{\mathrm{div}}(f) = \mathrm{div}(f)_{fin} + v_\infty(f) \cdot [F_\infty]$

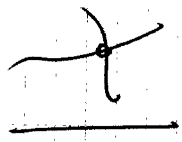
usual prime div assoc f

$$v_\infty(f) = -\int_{\mathcal{X}(\mathbb{C})} \log|f| \, \mu_{Ar}$$

$$\begin{tikzcd}
\widehat{\mathrm{Div}}(\mathcal{X}) \arrow[r] \arrow[d] & \widehat{\mathrm{Div}}(\mathcal{X}) / \widehat{\mathrm{div}}(\mathcal{X}) \arrow[d, equal] \\
\widehat{\mathrm{Pic}}(\mathcal{X}) & \widehat{\mathrm{Ch}}(\mathcal{X}) \arrow[l, dashed, "\exists !"', "\approx"]
\end{tikzcd}$$

How to define intersection pairing
$D_1, D_2 \in \widehat{\mathrm{Div}}(\mathcal{X})$

1) D_1 vertical D_2 Weil
we already know (Liu)



2) D_1 horizontal, $D_2 = [F_\infty]$

$$D_1 \cdot D_2 = \deg_{X_\eta}(D_{1\eta}) \qquad \text{(as if } F_\infty \text{ were an actual fiber)}$$

3) D_1 D_2 sections of $\mathcal{X} \to \mathrm{Spec}\, \mathbb{Z}$, $D_1 \neq D_2$

$$(D_1, D_2)_{Ar} = (D_1, D_2)_{fin} + (D_1, D_2)_{inf}$$

$(D_1, D_2)_{fin}$ usual number

$$(D_1, D_2)_{inf} = -\log G(D_{1,\mathbb{C}}, D_{2,\mathbb{C}})$$

<u>Thm</u> (Arakelov)

This induces a symmetric bilinear pairing s.t.

$$\widehat{\mathrm{Ce}}(\mathcal{X}) \times \widehat{\mathrm{Ce}}(\mathcal{X}) \longrightarrow \mathbb{R}$$