ARIYAN JAVANPEYKAR: Lecture 6 - Effective Shafarevich

Szpiro's Small pts Conj & effective Morde

§ Introduction

Theorem (Faltings 1983)

K #-field $\forall g \geq 2$ $\forall S$ finite set of finite places of K

Shaf. Conj [The set $\mathcal{M}_g(\mathcal{O}_{K,S}) = \{$smooth proper curves of genus g $/\mathcal{O}_{K,S}\}/_{\text{isom}}$ is finite

Mordell conj [$\forall$ $X/_K$ smooth proj curve of genus at least 2 th set $X(K)$ is finite

Let $\mathcal{X}/_{\mathcal{O}_{K,S}}$ be a proper model for X. Then

$$\mathcal{X}(\mathcal{O}_{K,S}) = X(K) \qquad \text{(relationship)}$$

Faltings: Shafarevich Conj $\xRightarrow{\text{Parshin}}$ Mordell Conj (USE KODAIRA'S TRICK)

§ Kodaira's Trick

Theorem (Kodaira)

Let $X/_{\mathbb{C}}$ be a smooth proper curve of genus ≥ 2

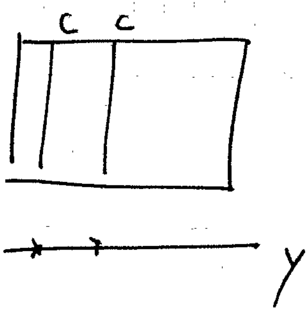
Then: there exists the following data

- $\begin{array}{c} Y \\ \downarrow \\ X \end{array}$ finite étale
- $g \geq 2$ (infact, $g \geq 3$)
- $Y \xrightarrow{\varphi} \mathcal{M}_g$ "non-trivial" i.e. finite fibers

i.e. $\varphi \in \mathrm{Hom}(Y, \mathcal{M}_g) = \mathcal{M}_g(Y) = \left\{ \begin{array}{c} S \\ \downarrow \varphi' \\ Y \end{array} \text{ family of sm. proper curves of genus } g \text{ over } Y \right\}$

φ NON-ISOTRIVIAL (= truly varying)

Non-Ex of φ: $C/_{\mathbb{C}}$ of genus g



$$\begin{array}{c} S \overset{\text{def}}{=} C \times Y \\ \big\downarrow \pi_2 \\ Y \end{array}$$

$S \to Y$ is TRIVIAL.

<u>Proof of Mordell Conjecture</u> (assuming Shafarevich Conj.)

choose T finite set of finite places of K such that

$$\begin{array}{ccc} & & Y \xrightarrow[\text{fibers}]{\text{finite}} \mathcal{M}_{g,K} \\ & & \downarrow \\ \operatorname{Spec} K & \xleftarrow[\text{proper}]{} & X \end{array}$$

finite étale

extends

$$\begin{array}{ccc} \operatorname{Spec} \mathcal{O}_{K,T} & \xleftarrow[\text{proper}]{} & \mathcal{X} \\ & & \big\uparrow \text{finite étale} \\ & & \mathcal{Y} \xrightarrow[\text{finite}]{} \mathcal{M}_{g, \mathcal{O}_{K,T}} \end{array}$$

$X(K)$ finite

$\Updownarrow$

$\mathcal{X}(\mathcal{O}_{K,T})$ is finite

Chevalley-Weil: suffices to prove

$\mathcal{Y}(\mathcal{O}_{K,T})$ finite $\xrightarrow[\text{fibers}]{\text{finite}}$ $\mathcal{M}_g(\mathcal{O}_{K,T})$

(for all K, T)

But $\mathcal{M}_g(\mathcal{O}_{K,T})$ is FINITE by SHAFAREVICH Conj ∎

<u>Proof of Kodaira's Trick</u>

C curve of genus ≥ 2

to show: $\exists$ $\begin{array}{ccc} & & X \\ & & \downarrow \pi \\ C & \leftarrow & C_1 \end{array}$ smooth non-isotrivial family of curves of genus at least two

$C_1 \to C$ finite étale

$p \in C$ choose
$$\begin{aligned} C &\longrightarrow \mathrm{Jac}(C) \\ Q &\longmapsto Q - P \end{aligned}$$

$$\begin{tikzcd} C' \arrow[r] \arrow[d, "\varphi", "2:1"'] & A \arrow[d, "2:1"] \\ C \arrow[r] & \mathrm{Jac}(C) \end{tikzcd}$$
def by a non-trivial 2-torsion pt

$\Gamma \subseteq C \times C'$

$\Gamma = C' \times_C C' = \{(\alpha, \beta) : \varphi(\alpha) = \varphi(\beta)\}$

Black box (1) $\quad k = \mathbb{C}$

$\exists \quad C' \xrightarrow{\psi'} \mathrm{Pic}^\circ C'_{/k} = J'$

associated to $\mathcal{O}_{C \times C'}(\Gamma - 2\Delta)$

$$\begin{tikzcd} C' \times C_1 \arrow[r] \arrow[d, "\mathrm{id} \times R"'] & C_1 \arrow[r] \arrow[d, "R"] \arrow[ddl, bend right, "\alpha"'] \arrow[dr, phantom, "\circlearrowright"] & J' \arrow[d, "\times 2"] \\ C' \times C' \arrow[r] \arrow[dr] & C' \arrow[r] \arrow[d, "\varphi"] & J' \\ & C & \end{tikzcd}$$
(multiplication-map)

$$\begin{tikzcd} \Gamma_1 \arrow[d] & \subseteq & C' \times C_1 \arrow[d] \\ \Gamma & \subseteq & C' \times C' \end{tikzcd}$$

Black box (2)

Γ_1 has a root

$$\begin{tikzcd} & X \arrow[d] \arrow[dr, bend left, "\pi"] & \\ \Gamma_1 \arrow[r, hook] \arrow[rr, bend right, "\text{étale}"'] & C' \times C_1 \arrow[r] & C_1 \end{tikzcd}$$
ramified precisely along Γ_1

$\Rightarrow \pi$ smooth $\quad$ (Γ_1 étale $/C_1$) because

Last thing to verify : $\pi : X \to C_1$ is non-isotrivial

Suppose $\exists P_1, \ldots \in C_1$ s.t. $C_0 \simeq X_{P_1} \simeq \ldots$ (for some curve $C_0 / \mathbb{C}$)

$$\begin{tikzcd} X_{P_1} \arrow[r, "2:1"] \arrow[rr, bend right] & C' \arrow[r] & C \end{tikzcd}$$
ramified precisely at $\alpha(P_1)$

$$\begin{tikzcd}
X_{p_1} \arrow[dr, "\sim"'] \arrow[ddr, bend right, "\text{ramified precisely at } \alpha(p_1)"'] & X_{p_2} \arrow[d, "\wr"] & X_{p_3} \arrow[dl, "\sim"] & \cdots \\
& C_0 \arrow[d, bend right] \arrow[d] \arrow[d, bend left] & \\
& C &
\end{tikzcd}$$

(all those maps are distinct) Ramified along different pts

$g(C) \geq 2$

Not possible by a Thm of De-Franchis / Severi $\square$

§ OPEN PROBLEMS

- Effective Shafarevich Conjecture
  $\forall K \ \forall S \ \forall g \geq 2$ the set $\mathcal{M}_g(\mathcal{O}_{K,S})$ is FINITE and effectively computable

- Effective Mordell Conjecture
  $\forall K \ \forall X/_K$ smooth proper of genus $g \geq 2$
  The set $X(K)$ is finite and effectively computable

"Thm" (Rémond) effective Shafarevich Conjecture $\Rightarrow$ effective Mordell

proof : show that Kodaira's Trick allows you to bound Heights $\square$

Rmk
At the moment, there is NOT one curve $X/_K$ s.t. $\forall L/_K$, $X(L)$ is finite and eff. computable
(i.e. we don't know eff. Mordell for a single curve)

Suggestion : construct explicitely $X/_{\mathbb{Q}}$ with $X \to \mathcal{M}_{3,\mathbb{Q}}$
then show eff. Shafarevich Conj for $\mathcal{M}_3$ only
($\to$ NOT KNOWN)

Chris Zaal (1996) constructed explicit X and explicit $X \longrightarrow \mathcal{M}_3$ $\quad (g(X) \approx 500)$

$$\mathcal{M}_3 = \underset{\text{hyperelliptic curves}}{\mathcal{H}_3} \cup \underset{\text{smooth quartic in } \mathbb{P}^2}{Q_3}$$

Thm 1 (J. 2016)

$\forall K, \forall S$ The set $Q_3(\mathcal{O}_{K,S})$ is finite and eff. computab

Thm 2 (J. 2015) w/ Von Kanel

$\forall K, \forall S$ The set $\mathcal{H}_3(\mathcal{O}_{K,S})$ is finite and eff computab

Remark: think of the analogous situation

$$\underset{\text{(not hyperbolic)}}{\mathcal{M} = \mathbb{P}^1} \qquad \underset{\text{(hyperbolic)}}{\mathcal{H} = \{0, 1, \infty\}} \qquad \underset{\text{(hyperbolic)}}{Q = \mathbb{P}^1 \setminus \{0, 1, \infty\}}$$

proof (of Thm 1)

$X \hookrightarrow \mathbb{P}^2$ smooth quartic curve

$$\begin{array}{c} S \\ \downarrow 2:1 \\ X \hookrightarrow \mathbb{P}^2 \end{array} \quad \text{ramified along } X \text{ precisely}$$

S smooth dP surface of degree two

$S \longrightarrow \mathbb{P}^2$ blow-up at 7 pts

$(0:0:1)$ $(0:1:0)$ $(1:0:0)$ $(1:1:1)$ P_5 P_6 P_7

$P_i = (\alpha_i : \beta_i : \gamma_i)$

"gen position", good reduction $\Longrightarrow \alpha_i \; \beta_i \; \gamma_i \in \mathcal{O}_{K,S}^*$

$P_i = (a_i ; b_i : 1)$ $\quad a_i = \frac{\alpha_i}{\gamma_i} \quad b_i = \frac{\beta_i}{\gamma_i}$

$(a_i - 1)(b_i - 1) \in \mathcal{O}_{K,S}^* \Rightarrow a_i \; b_i$ solve the unit eq $\square$

(Note: X is blow-up of $\mathbb{P}^2$ in $(0:0:1), (0:1:0), (1:0:0), (1:1:1), P_5, P_6, P_7$

§ Szpiro's small points Conjecture

Conj (Szpiro)

$\forall K \quad \forall S \quad \forall g \geq 2 \qquad \exists\, C(K,S,g) \in \mathbb{R}$ $\boxed{\text{EXPLICIT}}$

such that $\quad \forall X \in \mathcal{M}_g(\mathcal{O}_{K,S}) \quad \exists\, b \in X(\bar{K})$:

$$h(b) \leq C(K,S,g)$$

"Arakelov ht of b"

$\rightsquigarrow$ reduce this to Belyi degrees

- $\Delta(X) \leq 10^9 \deg_B(X)^7$
- $\exists\, b \in X(K)$ s.t. $h(b) \leq 10^9 \deg_B(X)^7$
- $h_{FAL}(X) \leq 10^9 \deg_B(X)^7$

Conjecture B $\quad \forall K \ \forall S \ \forall g \geq 2 \quad \exists\, C(K,S,g)$ explicit s.t.

$\forall X \in \mathcal{M}_g(\mathcal{O}_{K,S}) \qquad \deg_B(X) \leq C(K,S,g)$

$$\begin{tikzcd}
 & \text{Eff. Shafarevich} \arrow[dr, Rightarrow, "\text{Rémond}"] & \\
\text{Conj (B)} \arrow[ur, Rightarrow, "\text{J.-von Känel}"] \arrow[dr, Rightarrow, "\text{J.-von Känel}"'] & & \text{Eff. mordell} \\
 & \text{Szpiro's small point conjecture} \arrow[ur, Rightarrow, "\text{Szpiro}"'] &
\end{tikzcd}$$

RMK These implications are to be taken with a grain of salt

Proof Conj (B) for hyperelliptic curves

$\underset{hyp}{X} \xrightarrow{2:1} \mathbb{P}^1 \supset B$ "branch pts"

H = heights of B = $\max\{H(b) \mid b \in B\}$

$N = \# \bigcup_{b \in B} \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\{b\}$

Lily Khadgavi's eff. version of Belyi's thm

$$\deg_B(X) \leq 2(4N \cdot H)^{9N^3 2^N N!}$$

$$N \leq (\overset{\text{FIXED}}{2g+2})[\overset{\text{FIXED}}{K:\mathbb{Q}}]$$

$$C_R(B) = \left\{ \frac{\lambda_1 - \lambda_2}{\lambda_3 - \lambda_4} : \lambda_i \in B \right\} \quad \text{cross-ratios}$$

$$H \leq \max_{\lambda \in C_R(B)} \{ \text{Height}(\lambda) \}$$

$\displaystyle H(\lambda)_{\lambda \in C_R(B)} \leq$ Explicit constant, as λ solves the unit eq. in L, S'_L controlled in terms of k, S, g $\square$

Rmk This proves all conjectures [(B), Eff. Shaf, Szpiro's small pts Conj.] for Hyp. ell. curves.

But that's not enough to obtain eff. Mordell.