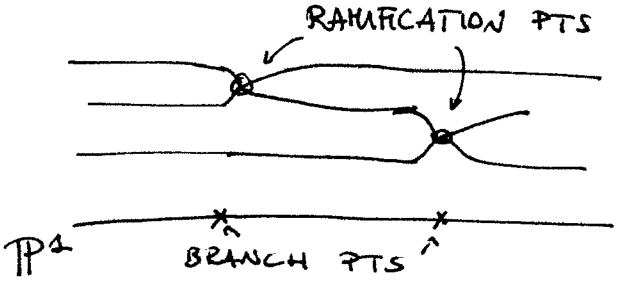
ARIYAN JAVANPEYKAR: Lecture 1 - BELYI's THEOREM

Belyi's Thm

$a, b \neq 0 \qquad a, b \in \mathbb{Z} \qquad \pi_{a,b}(x) = x^a (x-1)^b$

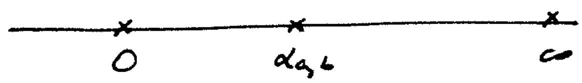


$\pi'_{a,b}(x) = a x^{a-1}(x-1)^b + b x^a (x-1)^{b-1}$

$\pi'_{a,b} = 0 \qquad x = 0 \ \vee \ x = 1 \qquad a(x-1) + bx = 0 \iff x = \frac{a}{a+b}$

$\pi_{a,b}(0) = \pi_{a,b}(1) = 0 \qquad \alpha_{a,b} = \pi_{a,b}\left(\frac{a}{a+b}\right)$

for $\pi_{a,b}$



only branch pts for $\pi_{a,b}$

Def A function $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ is a Belyi's map / function, morphism / dessin d'enfant if the number of branch points of π is at most 3.

THM (Belyi's Algorithms)
let $B \subset \mathbb{P}^1(\overline{\mathbb{Q}})$ finite. There exists a function
$\pi : \mathbb{P}^1 \to \mathbb{P}^1 \ / \mathbb{Q}$ s.t.

1) π is Belyi's function ramified only over $0, 1, \infty$
2) $\pi(B) \subseteq \{0, 1, \infty\}$

idea of proof: assume $B = \{0, 1, \infty, \lambda\}$ with $\lambda \in \mathbb{Q}$
choose $a, b \in \mathbb{Z} \setminus \{0\}$ s.t. $\lambda = \frac{a}{a+b}$
then $\pi = \pi_{a,b} / \pi_{a,b}(\lambda)$

Def A function $\pi : X \to \mathbb{P}^1$, X smooth proj connected curve/$\overline{\mathbb{Q}}$
is called a Belyi map if it ramifies only over 3 points.

Thm (Belyi $1 \Rightarrow 2$, Grothendieck $2 \Rightarrow 1$)

The following are equivalent

1) X can be defined over $\overline{\mathbb{Q}}$
2) $\exists\ \pi : X \to \mathbb{P}^1_{\mathbb{C}}$ Belyi map

Pf $1) \Rightarrow 2)$ $\exists\ f : X \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ finite morphism. Let B = branch of f.
Belyi's alg $\Rightarrow \exists\ \pi : \mathbb{P}^1_{\overline{\mathbb{Q}}} \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ Belyi and $\pi(B) \subseteq \{0, 1, \infty\}$
define $\pi' = \pi \circ f$

Rmk Any Belyi map can be given up to composing with some (actually 1) automorphism of $\mathbb{P}^1$ by a function with coefficients in $\overline{\mathbb{Q}}$

EXAMPLE ["$1 \Rightarrow 2$"]

$$\begin{array}{ccc} F(n) = \{x^n + y^n = z^n\} & n \geq 1 & /\mathbb{Q} \\ \downarrow \quad (x, y, z) & & \\ \quad\ \ \downarrow & \text{Belyi (degree } n^2\text{)} & \\ \mathbb{P}^1 \quad (x^n : z^n) & & \end{array}$$

EXAMPLE ["$2 \Rightarrow 1$"]

$\Gamma \leq SL_2(\mathbb{Z})$ finite index ex $\Gamma(N)$, $\Gamma_0(N)$, $\Gamma_1(N)$

$SL_2(\mathbb{Z}) \curvearrowright \mathbb{H}$ via Moebius transf so $\Gamma \curvearrowright \mathbb{H}$

$$\begin{tikzcd}
Y(\Gamma) := \Gamma \backslash \mathbb{H} \arrow[r, hook] \arrow[d] & X(\Gamma) \arrow[d] \arrow[dd, bend left] \\
X(1) := SL_2(\mathbb{Z}) \backslash \mathbb{H} \arrow[r, hook] \arrow[d, "\wr"] & X(1) \arrow[d, "\wr\, j"] \\
\mathbb{C} \arrow[r, hook] & \mathbb{P}^1
\end{tikzcd}$$

$X(\Gamma) \longrightarrow \mathbb{P}^1$ is a Belyi map

$\overset{2 \Rightarrow 1 \text{ Belyi thm}}{\Longrightarrow}$ $X(\Gamma)$ can be defined over some number field

Def $X / \overline{\mathbb{Q}}$ smooth proj connected $/ \overline{\mathbb{Q}}$

$$\deg_B(X) = \text{'Belyi's degree of } X\text{'}$$
$$= \min \{ \deg \pi \mid \pi \text{ is Belyi's on } X \}$$

(well defined by Belyi)

Examples: - $\deg_B (F(n)) \leq n^2$

- $\deg_B (X(\Gamma)) \leq [SL_2(\mathbb{Z}) : \Gamma]$

X : $y^2 + y = x^{2g+1}$ $\qquad g \geq 1$

the map $\pi : X \to \mathbb{P}^1$ $\quad \pi(x,y) = y$ is a Belyi map of degree $2g+1$ which is minimal

$$\deg_B(X) = 2g + 1$$

Properties

- $\deg_B(X) = 1 \iff X \simeq \mathbb{P}^1$
- $\deg_B(X) \geq 3$ Riemann - Hurwitz
- $\deg_B(X) = 3 \iff X$ genus one $\quad j(X) = 0$ ($1, 1728$
  $y^2 + y = x^3$

PROPOSITION let $d \in \mathbb{Z}_{\geq 1}$ then

$\{ X / \overline{\mathbb{Q}} : \deg_B(X) \leq d \}$ is finite set. "super Northcott property"

Pf Note $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\})$ is finitely generated
For all $d \in \mathbb{Z}_{\geq 1}$ $\quad \pi_1(-)$ has only ~~...~~ finitely many subgroups of index d

Galois theory $\Rightarrow$ the set $S = \{ V \downarrow \mathbb{P}^1 \setminus \{0,1,\infty\}$ finite étale of $\deg \leq d \}_{/\mathrm{Iso}}$ is FINITE!

Conclusion: $\{ X/\overline{\mathbb{Q}} \mid \deg_B(X) \leq d \}_{/\mathrm{Isom}} \twoheadleftarrow S$

$\square$

proof of Grothendieck – Weil

Let $X \xrightarrow{f} \mathbb{P}^1_{\mathbb{C}}$ be a Belyi's map with branch $= \{0,1,\infty$

If (*) $\mathrm{Aut}(\mathbb{C}) \cdot \{f\}$ is finite we are done.

To show (*) take $\sigma \in \mathrm{Aut}(\mathbb{C})$

$$f^\sigma : X^\sigma \longrightarrow \mathbb{P}^1 \quad \text{is again Belyi map ramif over } 0,1,\infty$$

$\deg f = \deg f^\sigma$

so $\{ f^\sigma : \sigma \in \mathrm{Aut}(\mathbb{C}) \} \subseteq \{$ Belyi map of degree $\leq \deg f \}$ $\Rightarrow$ *

is FINITE ∎

Consequence :

$$\mathcal{M}(\overline{\mathbb{Q}}) = \{ \text{curves} / \overline{\mathbb{Q}} \}_{/\text{iso}} = \coprod_{g \geq 0} \mathcal{M}_g(\overline{\mathbb{Q}})$$

any function $f : \mathcal{M}_g(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}$

then $\exists\, h : \mathbb{R} \to \mathbb{R}$ s.t. $f(X) \leq h(\deg_B(X))$

example

$$g : \mathcal{M}(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R}$$

$\to \deg (y^2 + y = x^{2g}$
$= 3$ minima

then $2g(X) + 1 \leq \deg_B(X)$

Future plan : look at
- Faltings height
- discriminant
- self intersection of dualas. sheaf