ARIYAN JAVANPEYKAR: Lect 4 : Discriminant bound

§ 4.1 Aim of this talk : explain the proof of

Thm (J, 2014) X smooth, proj., connected curve let $X/\overline{\mathbb{Q}}$ be a curve. The inequality

$$\Delta(X) \leq 10^9 \deg_B(X)^7 \quad \text{holds}$$

"Theorem" : any Arakelov invariant of $X/\overline{\mathbb{Q}}$ (eg $\Delta(X)$, h_{Fa} is bounded by an explicit polynomial in the Belyi degree of X

Recall : - $\deg_B(X) = \min \{ \deg \pi : \pi : X \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ Belyi map $\} \in$

- $\Delta(X)$: choose K numb. field $K \hookrightarrow \overline{\mathbb{Q}}$:

$\mathcal{X}/\mathcal{O}_K$ minimal semistable regular model

$$\Delta(X) = \frac{1}{[K:\mathbb{Q}]} \sum_{\substack{\mathfrak{p} \subseteq \mathcal{O}_K \\ \text{max ideal}}} \# \operatorname{Sing}(\mathcal{X} \otimes \overline{k(\mathfrak{p})}) \log(\# k($$

$k(\mathfrak{p}) :=$ residue field at

§ 4.2 Application

$\Gamma \leq SL_2(\mathbb{Z})$ congruence subgroup $k \in \mathbb{Z}_{>1}$

A modular form f of weight k for Γ has a q-expansion

$$f = \sum_{m=0}^{\infty} a_m(f) q^m$$

and is determined by $a_0(f), a_1(f), \dots, a_{k[SL_2(\mathbb{Z}):\Gamma]}$

Theorem (Couveignes - Edixhoven - Bruin - J., 2014)

Assume the Riemann Hypothesys for ζ-functions of nf's, there exists a probabilistic algorithm that, given

- $k \in \mathbb{Z}_{\geq 1}$
- $\Gamma \leq SL_2(\mathbb{Z})$ congruence
- K # field
- f modular form of weight k for Γ/K $\boxed{1/}$

- an integer $m \in \mathbb{Z}_{\geq 1}$ in factored form

computes $a_m(f)$ and whose expected running time is bounded by a POLYNOMIAL in the length of the input.

"proof" algorithm due to C - E - B

To show that algorithm runs in poly time it suffices to show $\Delta(X_1(N))$ is polynomial in N

(When p is a prime number s.t. $p^2 \mid N$ $\Delta(X_1(N))$ is more difficult to study)

$$\begin{array}{c} X_1(N) \\ \text{Belyi} \downarrow \text{degree} \leq N^2 \\ X(1) \end{array}$$

Thm $\Rightarrow$ $\Delta(X_1(N)) \leq 10^9 N^{14}$ $\square$

Other applications:

II Edixhoven - de Jong - Schepers Conjecture

Faltings height of a cover of curves

III Szpiro's small points conjecture for cyclic covers of prime degree (w/ von Känel)

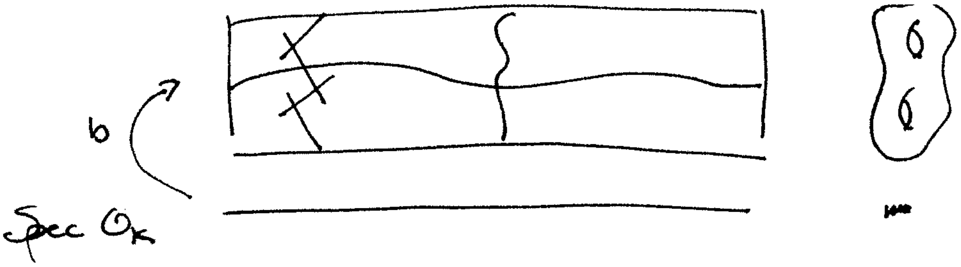
IV an effective version of the Hyperbolic isogeny theorem

§ 4.3 Assume $g(X) \geq 1$ (otherwise trivial $\Delta(\mathbb{P}^1)=0$)

Fix $b \in X(\overline{\mathbb{Q}})$ choose

K, $K \hookrightarrow \overline{\mathbb{Q}}$ $\quad \mathcal{X}$ minimal, regular, semi-st

$$\begin{array}{c} \mathcal{X} \\ \downarrow \curvearrowright b \\ \operatorname{Spec} \mathcal{O}_K \end{array}$$



Arakelov height of b : $(g \geq 2)$

$$h(b) = \frac{1}{[K:\mathbb{Q}]} \left(\mathcal{O}_{\mathcal{X}}(b), \omega_{\mathcal{X}/\mathcal{O}_K} \right)_{AR}$$

explicately : $s \in \omega_{\mathcal{X}/\mathcal{O}_K}$ non-zero rat'l section

assume $b \notin \operatorname{supp}(\operatorname{div}(s))$

$$[K:\mathbb{Q}] \, h(b) = \underbrace{(b, \operatorname{div}(s))_{\text{fin}}}_{\text{usual int. (length of local rings)}} + (b, \operatorname{div}(s))_{\text{inf}}$$

$$(b, \operatorname{div}(s))_{\text{inf}} = \sum_{\sigma : K \hookrightarrow \mathbb{C}} - \log \| s_\sigma \|_{AR} (b_\sigma)$$

assume $s = f \cdot dz$ $\quad z$ local coordinate of b in $X_{\mathbb{C}}$ R.S.

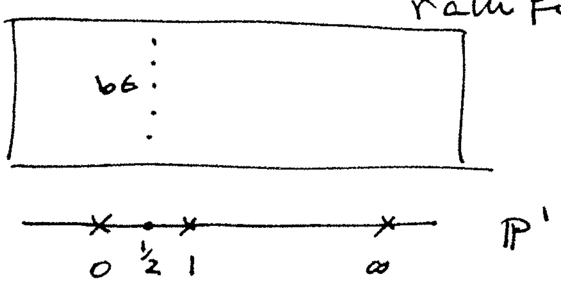
$$\log \| dz \| (b) := \lim_{a \to b} \left[g_{R_X}(a,b) - \log |z(a) - z(b)| \right]$$

Theorem (B)

$\forall\, b \in X(\overline{\mathbb{Q}})$ non-Weierstraß $\quad \Delta(X) \leq 100\, g^3\, h(b)$ + "analytic term"

Moral : to prove discriminant bound, find a small point

How to do this: Let $\pi : X \longrightarrow \mathbb{P}^1_K$ be a Belyi map ramified at $0, 1, \infty$



choose $b \in \pi^{-1}(1/2)$

Theorem (C) $\forall b \in \pi^{-1}\{\tfrac{1}{2}\}$ $\qquad h(b) \leq 10^7 \dfrac{\deg_B(X)^5}{g}$

Moral: Theorem (C) $\Rightarrow$ Main discriminant bound

to compute height we need a section: take $s = d\pi$

Theorem (D) $\forall b \in \pi^{-1}\{\tfrac{1}{2}\}$

(Arithmetic part) $\qquad (b, \operatorname{div}(d\pi))_{fin} \leq 10\,(\deg \pi)^3\,[K:\mathbb{Q}]$

(Analytic part) $\qquad (b, \operatorname{div}(d\pi))_{inf} \leq \dfrac{10^6 (\deg \pi)^5}{g}\,[K:\mathbb{Q}]$

Bounding Analytic part related to bounding Arakelov-Green's function

$$\begin{tikzcd}
X \arrow[r, phantom, "\underset{open}{\supseteq}"] \arrow[d] \arrow[dr] & V \arrow[d, "\substack{finite\\ étale}"'] & \mathbb{H} \arrow[l] \arrow[dl] \\
\mathbb{P}^1 \arrow[r, "\sim"', "\lambda"] & X(2) \arrow[r, phantom, "\underset{open}{\supseteq}"] & Y(2)
\end{tikzcd}$$

$V = \Gamma\backslash\mathbb{H}$

$\Gamma(2)/\{\pm 1\}$ is fundamental group

§4.4 Arithmetic part

$$\begin{tikzcd}
 & & \mathcal{X}' \arrow[dl, "\psi"', "\substack{proper\\ birat}"] \arrow[dr, "\substack{proper\\ birat}"] & \\
X \arrow[r, "\substack{norm.\\ of\ \mathbb{P}^1_{\mathcal{O}_K}\\ in\ K(X)}"'] \arrow[d] & \mathcal{X}^n \arrow[d, "\substack{finite\ flat\\ surj}"] & & \mathcal{X}^{min} \arrow[ddl] \\
\mathbb{P}^1_K \arrow[r] \arrow[dr] & \mathbb{P}^1_{\mathcal{O}_K} \arrow[drr] & & \\
 & \operatorname{Spec} K \arrow[r] & \operatorname{Spec} \mathcal{O}_K &
\end{tikzcd}$$

$\operatorname{div}(s) = K_{\mathcal{X}^{min}}$

$$(b, K_{\mathcal{X}^{min}})_{fin} \leq (b, K_{\mathcal{X}'})_{fin} \leq (b, K_{\mathcal{X}^n})_{fin}$$

$\omega_{\mathcal{X}'} = \omega_{\mathcal{X}^{min}} + E$ $\qquad$ $K_{\mathcal{X}'} = \psi^* K_{\mathcal{X}^n} + \sum c_i E$, $c_i \leq 0$

R.H. $\quad K_{\mathcal{X}^n} = \pi^* K_{\mathbb{P}^1_{\mathcal{O}_K}} + R = -2\pi^*\infty + R$

$$(b, K_{\mathcal{X}_{min}})_{fin} \leq (b, R)_{fin, \mathcal{X}} \underset{\uparrow}{\leq} (\tfrac{1}{2}, B)_{fin, \mathbb{P}^1_{\mathcal{O}_K}} \quad (\deg \pi)$$

proj. formula

B = branch divisor

$$\begin{array}{ccc} \mathcal{X} & & \coprod_{i \in I} \sum_{j \in I} D_{ij} \\ \downarrow & & / \\ \mathbb{P}^1_{\mathcal{O}_K} & \hookleftarrow & D = \sum_{i \in I} D_i \end{array}$$

$$\mathcal{O}_{\mathbb{P}^1_{\mathcal{O}_K}, D_i} \subseteq \mathcal{O}_{\mathcal{X}, D_{ij}}$$

DVR $\qquad$ DVR

(of characteristic zero with imperfect residue field possibly.)

PROPOSITION (Lenstra's generalisation of Dedekind's discr Thm

A complete DVR $\qquad K = \mathrm{Frac}(A)$

L/K finite field ext $\qquad [L : K] = N$

B integral closure of A in L

$$\begin{array}{ccc} A & \subseteq & B \\ \cap & & \cap \\ K & \subseteq & L \end{array}$$

note: B is DVR

Assume $\mathrm{char}(A) = 0$,

let $\mathcal{D}_{B|A}$ different ideal $\subseteq B$

$$\mathcal{D}_{B|A}^{-1} = \{ x \in L : \mathrm{Tr}(xB) \subseteq A \}$$

think $N = \deg \bar{\pi}$

The valuation of $\mathcal{D}_{B/A}$ is at most $N - 1 + N \cdot \mathrm{ord}_A(N)$)

Remark The residue field is not necessarily perfect

(more generally $k_A \subseteq k_B$ might not be separable)

In the tame case ramification index $= e$

and $e - 1 \leq N - 1$

proof (Lenstra)

$x \in A$ uniformizer, $\mathrm{ord}_A(x) = 1$

$y := \frac{1}{Nx} \in L$ $\qquad \mathrm{Tr}_{L/K}(y) = \frac{1}{x} \notin A$ i.e. $y \notin \mathcal{D}^{-1}_{B/A}$

$\mathcal{D}^{-1}_{B/A} \subsetneq yB \implies \frac{1}{y}B = (Nx)B \subsetneq \mathcal{D}_{B/A}$

$\implies \mathrm{ord}(\mathcal{D}_{B/A}) \leq \mathrm{ord}_B(Nx) - 1 = e\, \mathrm{ord}_A(Nx) - 1$

$= e\, \mathrm{ord}_A(N) + e - 1$

with $m_A B = m_B^e$.

Since $e \leq N$, this concludes the proof. $\square$