ARIYAN JAVANPEYKAR : lecture 5 – SHAFAREVICH CONJECTURE

K number field

S finite set of $\overset{\text{finite}}{}$places of K $= \{\mathfrak{p}_1 - \mathfrak{p}_N\} \subseteq \operatorname{Spec} \mathcal{O}_K$

$\operatorname{Spec}(\mathcal{O}_K) \setminus S = \operatorname{Spec} \mathcal{O}_{K,S} = \operatorname{Spec} \mathcal{O}_K[S^{-1}]$

<u>Example</u>/Thm (Hermite – Minkowski)

Let $d \in \mathbb{Z}_{\geq 1}$, the set of number fields L/K of degree d ramified only over S is finite, i.e. the set of (isomorphism classes) of finite étale covers of $\operatorname{Spec} \mathcal{O}_{K,S}$ of degree d is finite

Ex/<u>Thm</u> (Shafarevich, 1962 ICM)

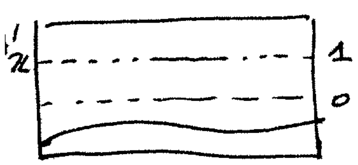
The set of elliptic curves E/K with good reduction outside S is finite, i.e the set of elliptic curves $\mathcal{E}/\mathcal{O}_{K,S}$ is finite. [N.B. One should work "up to isomorphism". We will omit this for the sake of brevity

<u>Example</u> / Thm (Faltings '83, Shafarevich Conj 1962 ICM)

Let $g \geq 1$. The set of g-dim'l principally polarized abelian varieties over K with bad reduction only at S is finite. Equivalently the set of g-dim'l pp ab. varieties schemes over $\mathcal{O}_{K,S}$ is finite.

<u>Example</u> / Thm

The set of solutions to the unit equation $x+y=1$ in $\mathcal{O}_{K,S}$ is finite, i.e. $\{(x,y) \in \mathcal{O}_{KS} \times \mathcal{O}_{KS} : x+y=1,\ x,y \in \mathcal{O}_{KS}^*\}$



$= (\mathbb{A}^1_{\mathbb{Z}} - \{0,1\})(\mathcal{O}_{KS})$ is finite

PHILOSOPHY : The set of "objects of fixed type" over $\mathcal{O}_{K,S}$ is finite, in general

"Counterexample ①" set of all integers in $\mathcal{O}_{K,S}$ is <u>NOT</u> finite

<u>Thm</u> (Faltings)

Fix $d \in \mathbb{Z}_{\geq 1}$, ℓ prime number, $w \in \mathbb{Z}_{\geq 1}$

The set of semisimple ℓ-adic Galois representations

$$Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_d(\mathbb{Q}_\ell)$$

of dimension d and weight w which are unramified outside $S \cup \{\ell\}$ is finite.

<u>Thm</u> (Borel - Serre)

$G/\mathcal{O}_{K,S}$ affine group scheme of finite type.

The set of G-torsors over $\mathcal{O}_{K,S}$ is finite.

Aim: give conceptual explanation of these phenomena using Lang - Vojta Conjecture

§ <u>Lang - Vojta CONJECTURE</u>

$X/\mathbb{C}$ smooth q.p. variety

~~Def~~ X is <u>(Brody) hyperbolic</u> if all holomorphic maps

$$\mathbb{C} \longrightarrow X^{an} \quad \text{are constant}$$

Examples: • $\mathbb{P}^1_\mathbb{C}$, $\mathbb{A}^1_\mathbb{C}$ are NOT hyperbolic $\quad \mathbb{C} \to \mathbb{A}^1(\mathbb{C}) \subseteq \mathbb{P}^1_\mathbb{C}$

• $\mathbb{G}_{m,\mathbb{C}} = \mathbb{A}^1_\mathbb{C} \setminus \{0\}$ $\quad \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* = \mathbb{G}_{m,\mathbb{C}}^{an}$

so $\mathbb{G}_m$ is NOT hyperbolic

Ex X connected $\dim X = 1$ $g(X) = 1$

$$\mathbb{C} \xrightarrow[\text{cover}]{\text{univ}} X^{an} = \mathbb{C}/\Lambda \quad \text{not hyperbolic}$$

Theorem (Picard) $\dim X = 1$ X connected

X hyperbolic $\iff$ $X \neq \mathbb{P}^1_{\mathbb{C}}, \mathbb{A}^1_{\mathbb{C}}, \mathbb{G}_m$, ell curve

proof: $X \neq \mathbb{P}^1, \mathbb{A}^1, \mathbb{G}_m$, ell curve

$$\begin{tikzcd} & \tilde{X}^{an} \arrow[d] \\ \mathbb{C} \arrow[ur, dashed, "\exists!"] \arrow[r, "\text{holom}"'] & X \end{tikzcd}$$

univ cover $\tilde{X}^{an} \simeq \mathbb{H} \subseteq \mathbb{C}^N$ "Bounded"

$\mathbb{C}$ simply conn

(The lift $\mathbb{C} \dashrightarrow \mathbb{H}$ is unique up to choice of base point.)

$\rightsquigarrow$ holom functions; all constant by Liouville $\square$

Remarks

1. "Chevalley - Weil" Let $X \to Y$ be finite étale then X hyperbolic $\iff$ Y hyperbolic

$$\begin{tikzcd} & X \arrow[d] \\ \mathbb{C} \arrow[ur, dashed, "\exists!"] \arrow[r] & Y \end{tikzcd}$$

Lang - Vojta Conjecture

If $\mathcal{X}/\mathbb{Z}$ finite type scheme, s.t. $\mathcal{X}^{an}_{\mathbb{C}}$ is smooth q-p hyperbolic variety,
Then $\forall K, \forall S$ $\mathcal{X}(\mathcal{O}_{K,S})$ is finite.

Ex $X = \mathbb{P}^1, \mathbb{A}^1, \mathbb{G}_m$, ell curve

$\mathbb{P}^1(\mathbb{Z})$ is infinite $\Rightarrow$ $\mathbb{P}^1$ is NOT hyperbolic ✓

$\mathbb{A}^1(\mathbb{Z})$ is infinite $\Rightarrow$ $\mathbb{A}^1$ is NOT hyperbolic ✓

$\mathbb{G}_m(\mathbb{Z}) = \{\pm 1\}$ but $\mathbb{G}_m(\mathbb{Z}[\frac{1}{2}])$ is infinite $\Rightarrow$ $\mathbb{G}_m$ NOT hyp ✓

$E/\mathbb{Q}$ "$\mathrm{rk}(E) = 0$" but choose K s.t. $\#E(K) = \infty$

(rat pts = integral pts if E proper) $\Rightarrow$ E NOT hyp ✓

Thm (Faltings 1983, Siegel, L-V in dim 1)

$X/\mathbb{Z}$ f.t. $X_{\mathbb{C}}$ is hyperbolic and $\dim X_{\mathbb{C}} = 1$

then $X(\mathcal{O}_{K,S})$ is finite

Recall: "Shafarevich Thm" (ell curves) Chevalley-Weil

moduli of ell-curves $/\mathbb{Z}[\frac{1}{2}] \xleftarrow[\text{unramified}]{6:1 \text{ FINITE ÉTALE}} \mathbb{P}^1_{\mathbb{Z}[\frac{1}{2}]} \setminus \{0,1,\infty\}$

$\{$ ell curves $/\mathcal{O}_{KS}$ $\}$ $\xleftarrow{\text{legendre}}$ $\{$ sol to unit eq in $\mathcal{O}_{K,S}$ $\}$

in Legendre form $\qquad \lambda \quad 1-\lambda$

$y^2 = x(x-1)(x-\lambda)$ $\qquad$ FINITE by Siegel

Lang-Vojta + Chevalley-Weil $\Rightarrow$ Shafarevich's Thm

Rem: "Lang-Vojta $\Longrightarrow$ Shafarevich Conjecture"

$\mathcal{A}_g$ = moduli of p.p. abelian varieties of dim g $/\mathbb{Z}$

To show Lang-Vojta $\Longrightarrow \mathcal{A}_g(\mathcal{O}_{KS})$ is finite

$\qquad\qquad \| \qquad \{$ p.p. ab schemes $/\mathcal{O}_{K,S}$ of dim g $\}$

one option is consider

$$\mathcal{A}_g^{[N]} = \{ (A, \varphi) : A \in \mathcal{A}_g \quad \varphi \text{ full level } N \text{ structure i.e. } \varphi : A[N] \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^{2g} \}$$

$\mathcal{A}_g$ is NOT a scheme, but $\mathcal{A}_g^{(N)}$ is a scheme ($N \geq 3$)

and $\quad \mathcal{A}_g^{[N]} \longrightarrow \mathcal{A}_g$ finite étale

Lemma $\mathcal{A}_{g,\mathbb{C}}^{[N]}$ with $N \geq 3$ is hyperbolic proof

$$\begin{tikzcd} & \mathbb{H}_g \subseteq \mathbb{C}^N \arrow[d, "\text{univ cover}"] \\ \mathbb{C} \arrow[r, "\text{holom}"'] \arrow[ur, dashed, "\exists!"] \arrow[ur, bend right=60] & \mathcal{A}_{g,\mathbb{C}}^{[N]} \end{tikzcd}$$

bounded

constant by Liouville □

Want To prove : $\mathcal{A}_g(\mathcal{O}_{K,S})$ is finite

$$\mathcal{A}_g(\mathcal{O}_{K,S}) \subseteq \bigcup_{\substack{L \text{ nf's} \\ \mathcal{O}_{L,S_L} \text{ étale over } \mathcal{O}_{K,S} \\ \text{of degree } \leq 3^{2g}}} \mathcal{A}_g^{[3]}(\mathcal{O}_{L,S_L})$$

$$\begin{tikzcd} \operatorname{Spec} \mathcal{O}_{L,S_L} \arrow[r] \arrow[d] & \mathcal{A}_g^{[3]} \arrow[d, "\text{finite étale degree } 3^{2g}"] \\ \operatorname{Spec} \mathcal{O}_{K,S} \arrow[r] & \mathcal{A}_g \end{tikzcd}$$

but $\{L \text{ nf's } \mathcal{O}_{L,S_L} \text{ étale over } \mathcal{O}_{K,S} \text{ of deg} < 3^{2g}\}$ is finite by Hermite

$\Rightarrow \bigcup \mathcal{A}_g^{[3]}(\mathcal{O}_{L,S_L})$ finite by Lang-Vojta

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§ Shafarevich Conjecture for smooth hypersurfaces

J. with D. Loughran

Thm ① (J.- Loughran)

Assume Lang-Vojta Conjecture

Fix $d \geq 3$, $N \geq 2$, then the set of smooth hypersurfaces of degree d and dim N in $\mathbb{P}^{N+1}_{\mathcal{O}_{K,S}}$ is finite

proof:

STEP A : add level structure $\mathcal{C}^{(N)} \xrightarrow{\text{fin. étale}} \mathcal{C}$

STEP B : $\mathcal{C}^{[N]}$ is hyperbolic (infinitesimal Torelli)

STEP C : use Chevalley - Weil □

Thm 2 Fix $3 \le d \le 6$ $d \neq 5$

The set of smooth hypersurfaces of degree d in $\mathbb{P}^3_{\mathcal{O}_{K,S}}$ is finite

proof

$d = 3$ Siegel's thm (Scholl ') dP surfaces

$d = 4$ Y. André K3 surfaces "KUGA-SATAKE" Faltings Theorem

$d = 5$? (Work in progress)

$\boxed{d=6}$

~~fo~~

$$\left\{ \begin{array}{c} \text{smooth sextic} \\ \text{surfaces } X \subseteq \mathbb{P}^3_{\mathbb{C}} \end{array} \right\} \overset{y^2 = f}{\cong} \left\{ \begin{array}{c} Y \\ \downarrow 2:1 \\ X \hookrightarrow \mathbb{P}^3 \end{array} \text{ ramified along } X \right\}$$

$$\cong$$

$$\left\{ \text{FANO smooth threefolds } h^{2,1}(Y) = 52 \right\}$$

$H^3(Y) = 0 \quad 52 \quad 52 \quad 0$

$$\begin{array}{ccc} \{\text{smooth sextic surfaces}\} & & \{\text{FANO smooth threefolds}\} \\ & \underset{\text{finite fibers}}{\searrow} \quad \underset{\substack{\text{Infinitesimal torelli} \\ \Downarrow \\ \text{finite fibers}}}{\swarrow} & \\ & \mathcal{A}_{52,\mathbb{C}} & \end{array}$$

Faltings $\Rightarrow$ $\mathcal{A}_{52,\mathbb{C}}(\mathcal{O}_{K,S})$ finite . This is the "idea" of the proof. $\square$